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Master Thesis

Integrated Vehicle Control using Active Front Steering and Active Differential

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Abstract

Active differential is an interesting actuator for vehicle dynamics, which, providing asymmetric left / right traction, modifies the vehicle cornering behavior. Active front steering is indeed a high influence actuator for vehicle dynamics, which provides additional steering angle over the driver defined one. Both actuators are in production in high class vehicles, typically combined with brakes based stability controller.

This work wants to evaluate the possibility of combining active differentials with active front steering actuators in an integrated controller to guarantee vehicle stability and limit handling capability, with limited need of the brakes based controller support.

Introduction

Integrated Control Systems

Software will determine future vehicle performance to a great extent. Publications quote that 90% of future innovations will be directly linked with electronics. From this, 80% are said to be SW driven.

The vehicles of today are already equipped with many electronically controlled systems. A most popular system is the so-called ESP System. This system tries to stabilize the vehicle in case of loss of stability in acting on the engine and brakes.

If the system is active the engine control unit has to arbitrate several requests, coming from the drivers' gas pedal and from the ESP system; in case of automatic gearbox, an additional request can be observed. These sources of engine requests can conflict and therefore highly influence the driver's perception of the car. As more and more systems are introduced in the vehicle, it will furthermore be extremely complicated to capture all situations and transients, which can be occur.

A solution can be to introduce a hierarchical control structure where all control commands pass one core algorithm. With this concept, Ford Motor Company makes sure that the driver of the vehicle experiences an excellent, clearly defined, robust and reproducible behavior of the vehicle, with advantages as reusability and robustness of the system.

So, as indicated above, improved behavior can only be achieved by system integration. "Passive Systems" have been substituted, in the latest years, by "Stand Alone Active Systems"; this term includes systems loosely "integrated", which is currently standard in today's vehicles. Qualitative performance of both these system configurations is exceeded by the "Integrated Active Systems" one. By this, full system integration with closed loop control of all actuators in a coordinated way is meant. Only this will enable to fulfill the future vehicle dynamics demands.

Integrated Vehicle Control

Integrated vehicle control is the control of vehicle functions while taking vehicle subsystem interactions and driver-vehicle interactions into account. Vehicle functions are separately identifiable aspects of vehicle behavior or response, e.g., longitudinal acceleration. Vehicle subsystems are the physical devices which contribute to realizing the various vehicle functions, e.g., the powertrain. A high-level characterization of an integrated vehicle control system includes the following points:

1. The driver is considered not only as a source of manual control inputs, but as a dynamic subsystem of the vehicle which interacts with the other vehicle Subsystems.

2. The integrated control system coordinates the vehicle subsystems to achieve the desired responses for the vehicle functions as closely as possible and within the physical limits of the vehicle hardware.

3. The integrated control system achieves (1) and (2) in a way that is beneficial, pleasing and safe for the driver and passengers.

To achieve any form of electronic control in a vehicle, it must be endowed with the appropriate transducer set. Such a transducer set includes sensors, which supply information about the vehicle to the electronics, and actuators which receive control information from the electronics and supply energy to the vehicle. With the exception of aerodynamic forces, the primary forces affecting the motion of the vehicle are the longitudinal and lateral forces at the four tyre-road contact patches; these forces in turn depend on the normal forces appearing there. It is evident, then, that the tyre-road contact patches are interaction points for braking, steering, traction, and suspension.

Since a major goal of integrated vehicle control is the control of vehicle motion, the control of the longitudinal and lateral forces at the road-tyre contact patch is essential.

The key element of integrated vehicle control is that the behavior of the various vehicle subsystems is coordinated. The coordination of vehicle subsystems consists of trading off the performance levels of subsystems against each other, whenever these are in conflict, and more importantly, of getting the subsystems to behave as cooperatively as possible in performing the desired vehicle functions.

In short, by integrated vehicle control we mean the harmonious orchestration of vehicle subsystems such that the vehicle performs its functions to the maximum satisfaction of the driver. We do not mean the packaging and organization of vehicle electronics, nor do we mean the various approaches to inter-processor communication. These issues are extremely important, however, for determining how integrated control is implemented on a vehicle.

Chapter 1

Modeling

The vehicle motion can be in general described as a rigid body moving in the free space, therefore with 6 dof, connected with the ground surface through tyres and compliants, which give to the complete model high non-linear behavior and high coupling effects.

The actuators for this application are:

- active front steer (AFS), which can force an incremental steer angle on top of the driver's input, independently from this. The control is then actuated through the front axle tyres characteristic;
- active differential (AD), which can adjust the distribution of torque in the rear axle, usually to compensate undesirable handling behaviors like understeer. The control is then actuated through the rear axle tyres characteristic.

The goal of the present work is to control a 6 dof Black Box model, developed in Matlab-Simulink Environment by the Vehicle Dynamics Department of Ford Motor Company. It represents the behavior of a Jaguar S-Type in a very realistic way. It is called Black Box because its equations are hidden, so we can consider them unknown.

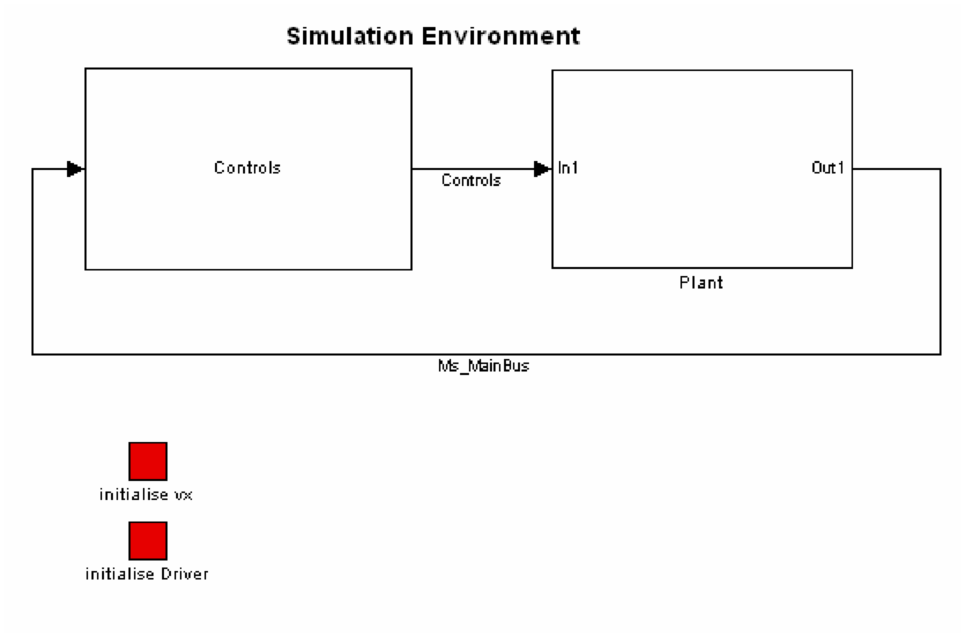


Figure 1.1: Black Box Vehicle Model

1.1 Working hypotheses

The plant complexity can be mitigated considering the following reasonable assumptions:

- the vehicle moves on a horizontal plane; in particular, this plane is an asphalted dry road;
- longitudinal velocity is little variable, without quick braking/accelerating actions; so vehicle shaking/pitch motions can be completely disregarded;
- the vehicle has stiff suspensions; so vehicle roll can be disregarded as well;
- as a consequence of the previous assumptions, the height of each point of the vehicle is kept constant; so the vehicle motion is a plane one;

- the steering system is perfectly rigid, so that the angular position of the front wheels is uniquely determined by the steering wheel position;
- the wheels masses are much lower than the vehicle one, so the steering action does not affect the position of the centre of mass of the entire vehicle;
- the vehicle takes large radius bends; that is to say that the curve radius of such bends is much higher than the vehicle width;
- the road wheel angles are "small" (less than 10°-15°); this assumption is combined with the previous one (large radius bends);
- with regard to the other forces acting on the vehicle, the aerodynamic resistance and the wind lateral thrust are not considered.

As a result of these hypotheses, we get a three degrees of freedom model.

1.2 Vehicle equations

$$m(\dot{v}_x - v_y v_\psi) = (F_{xfl} + F_{xfr})\cos\delta + F_{xrl} + F_{xrr} - (F_{yfl} + F_{yfr})\sin\delta \quad (1.1)$$

$$m(\dot{v}_y + v_x v_\psi) = (F_{yfl} + F_{yfr})\cos\delta + F_{yrl} + F_{yrr} + (F_{xfl} + F_{xfr})\sin\delta \quad (1.2)$$

$$\begin{aligned} J_z\dot{v}_\psi &= (F_{yfl} + F_{yfr})\cos(\delta)l_f - (F_{yrl} + F_{yrr})l_r + (F_{xfl} + F_{xfr})\sin(\delta)l_f + \\ &\quad -(F_{xfl} - F_{xfr})\cos(\delta)d - (F_{xrl} - F_{xrr})d + (F_{yfl} - F_{yfr})\sin(\delta)d \end{aligned} \quad (1.3)$$

$$\alpha_{fj} = \delta - \arctan\left(\frac{v_y + l_f v_\psi}{v_x \mp d v_\psi}\right) \quad (1.4)$$

$$\alpha_{rj} = -\arctan\left(\frac{v_y - l_r v_\psi}{v_x \mp d v_\psi}\right) \tag{1.5}$$

$$k_{rj} = \frac{\omega_{rj} R_w - (v_x \mp d v_\psi)}{v_x \mp d v_\psi} \tag{1.6}$$

$$F_{xij} = F_{xij}(\alpha_{ij}, F_{zij}, k_{ij}, \mu) \tag{1.7}$$

$$F_{yij} = F_{yij}(\alpha_{ij}, F_{zij}, k_{ij}, \mu) \tag{1.8}$$

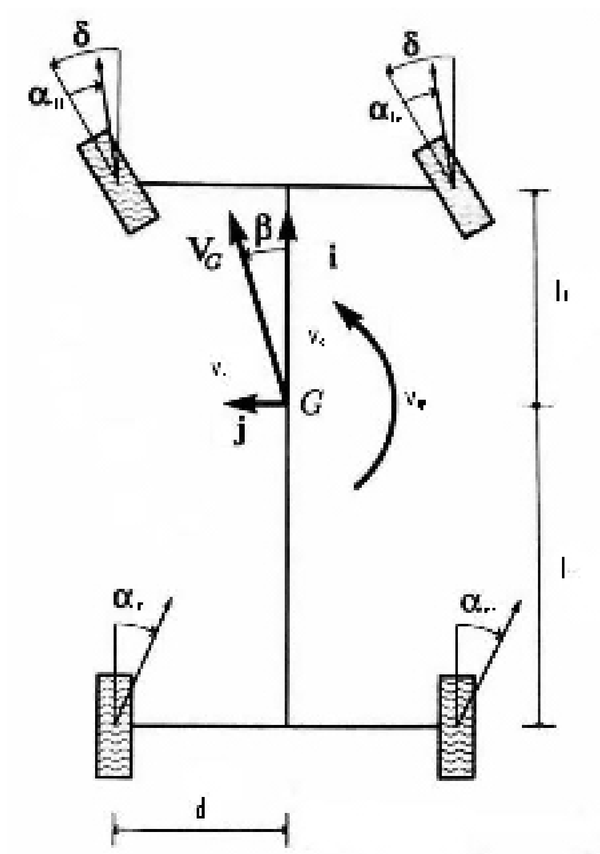
$$F_{zij} = F_{zij}(\phi, \theta) = F_{zij}(0, 0) = \text{cost}_i \tag{1.9}$$

$$\delta = \delta_C + \delta_D \tag{1.10}$$

where:

m	vehicle mass (kg)
J_z	vehicle inertia momentum (kg m^2)
v_x	vehicle longitudinal velocity (m/s)
v_y	vehicle lateral velocity (m/s)
v_ψ	vehicle yaw velocity (rad/s)
δ	road wheel angle (rad)
δ_C	controller road wheel angle component (rad)
δ_D	driver road wheel angle component (rad)
l_f, l_r	front and rear vehicle length (m)
d	half of the track length (m)
F_{xij}	tyre longitudinal force (N) (i=front, rear; j=left, right)
F_{yij}	tyre lateral force (N) (i=front, rear; j=left, right)
F_{zij}	tyre vertical load (N) (i=front, rear; j=left, right)
α_{ij}	tyre slip angle (rad) (i=front, rear; j=left, right)
k_{ij}	tyre longitudinal slip (i=front, rear; j=left, right)

ω_{ij} wheel angular velocity (rad/s) (i=front, rear; j=left, right)
R_w wheel radius (m)
$\mu = 1$ tyre-road friction coefficient
$\phi = 0$ vehicle roll angle (rad)
$\theta = 0$ vehicle pitch angle (rad)



Vehicle Model

1.3 Wheels equations

The bicycle model is extended modeling the wheels behavior according to the following equations:

$$J_w \, \dot{\omega}_{ij} = T_{ij} - R_w F_{xij} \tag{1.11}$$

$$T_{ij} = T_{Tij} - T_{Bij} \tag{1.12}$$

$$T_{Tfj} = 0 \tag{1.13}$$

$$T_{Trl} = T_{roll} - T_{AD} \tag{1.14}$$

$$T_{Trr} = T_{roll} + T_{AD} \tag{1.15}$$

where:

J_w wheel inertia momentum (kg m^2)

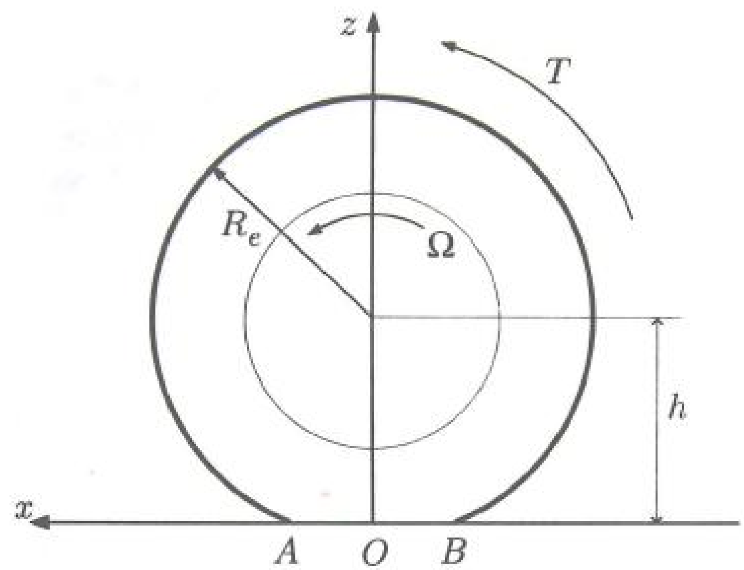
T_{ij} wheel torque (Nm) (i=front, rear; j=left, right)

T_{Tij} wheel traction torque (Nm) (i=front, rear; j=left, right)

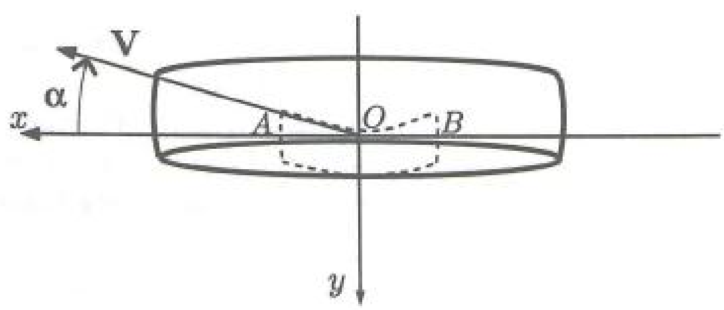
T_{Bij} wheel braking torque (Nm) (i=front, rear; j=left, right)

T_{roll} wheel free rolling torque (Nm)

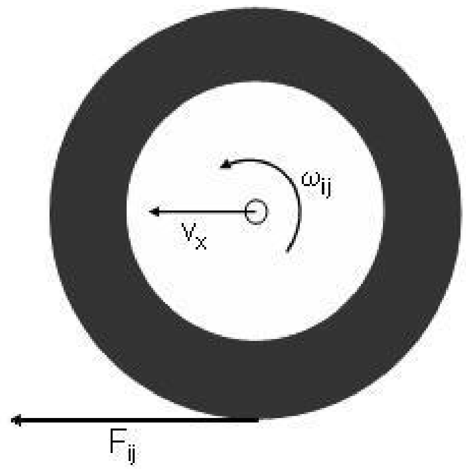
T_{AD} wheel differential torque (Nm)



Wheel coordinate system



Wheel coordinate system



Wheel model

1.4 Axis and coordinate systems

Vehicle dynamics models are typically generated using right-handed axis systems and coordinate systems. The axis orientation (ISO 8855) has X forward, Z up, and Y pointing to the left-hand side of the vehicle. This convention is recommended for several reasons. For instance, plots of Y vs. X show a top view of vehicle trajectories; vertical tire forces are always positive; and wheel spin rates are positive for forward vehicle speeds.

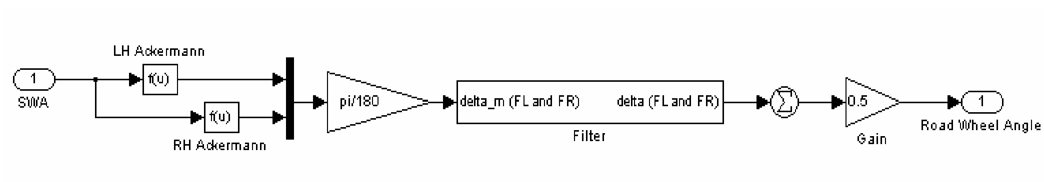
Alternative right-handed systems can be used, so long as X is longitudinal, Y is lateral, and Z is vertical. In our convention, positive yaw implies a left-hand turn.

The vehicle axis system $(x, y, z; G)$ is used the most for vehicle-level definitions. It has an X axis (unit vector i) that is pointing forward as the vehicle longitudinal axis, a Y axis (unit vector j) pointing to the left-hand side of the vehicle, a Z axis (unit vector k) that is parallel to the gravity vector. G is the centre of mass of the vehicle.

The wheel axis system is used the most for tyre-level definitions and for tyre forces. The origin point is the center of tire contact, the Y axis (unit vector j) is parallel with the spin axis of the wheel, the Z axis (unit vector k) is normal to the road and the X axis (unit vector i) is perpendicular to the Y-Z plane. The orientation is chosen to have a right-handed axis system.

1.5 Active Front Steering

The steering system dynamics is represented in the following picture



Steering system

The first section of this block is instantaneous, according to the following expressions

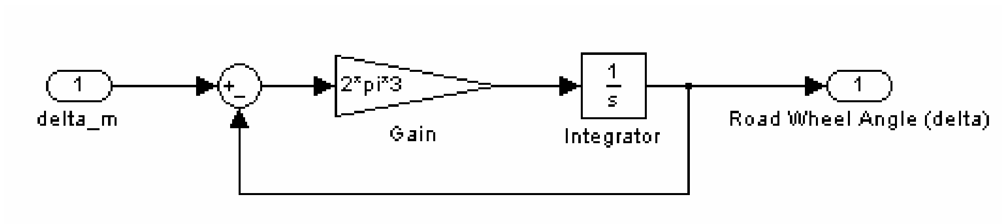
$$(0.06083 * u) - (4 * 10^{-6} * u^2) + (6 * 10^{-8} * u^3) - (4 * 10^{-11} * u^4)$$

for the front left wheel and

$$(0.06084 * u) + (4 * 10^{-6} * u^2) + (6 * 10^{-8} * u^3) + (3 * 10^{-11} * u^4)$$

for the front right one.

The second section of the steering system block is a low-pass filter



Steering system dynamics

whose input will be called δ_M in the model equations. This filter cancels possible quick steering oscillations.

The Active Front Steering (AFS) is an actuator which, according to the equation 1.10, forces an incremental steer angle on top of the driver's input, independently from this. We will not describe this actuator in detail. However, a reasonable assumption is about limiting the wheel angle correction to values of 5-10°. For higher values of correction, the actuator action is considered to be too invasive for the driver's perception of car.

1.6 Active Differential

A vehicle has wheels rotating at different speeds, especially when turning corners. The (passive) differential is designed to drive a pair of wheels with equal force, while allowing them to rotate at different speeds. In vehicles without a differential, such as karts, both driving wheels are forced to rotate at the same speed, usually on a common axle driven by a simple chain-drive mechanism. When cornering, the inner wheel travels a shorter distance than the outer wheel, resulting in the inner wheel spinning and/or the outer wheel dragging. This results in difficult and unpredictable handling, damage to tires and roads and strain on, and possible failure of the entire drive train. A (passive) differential is a device, usually consisting of gears, that allows each of the driving wheels to rotate at different speeds, while supplying equal torque to each of them.

The *electronically-controlled active differential* is a relatively new technology. A computer uses inputs from multiple sensors, including yaw rate, steering angle, and lateral acceleration and adjusts the distribution of torque to compensate for undesirable handling behaviors like understeer. In this work, the active differential is simply considered as a device that allows the driving wheels to have different torques besides different speeds. The difference of torque between left and right driving wheels is the AD degree of freedom for the control system, and no further assumptions are made about that.

The consequence of using an active differential is the need for the extension of a simple bicycle model through wheels equations to couple the vehicle subsystem and the wheels subsystem.

1.7 The system theory point of view

From the point of view of system and control theory, the vehicle and wheels equations describe a model with 7 state variables, 2 controlled inputs, 5 not controlled inputs and 7 outputs (measurable quantities).

- State variables:
  - vehicle longitudinal velocity v_x;
  - vehicle lateral velocity v_y;
  - vehicle yaw velocity v_ψ;
  - wheel angular velocity for each wheel ω_{ij}.
- Controlled inputs:
  - the rear axle right-left difference of torque T_{AD} (Active Differential)

  - the controlled road wheel angle component δ_C (Active Front Steering)

- External inputs:

  - the driver road wheel angle component δ_D;
  - wheels braking torques T_{Bij}.

- Outputs:

  - lateral acceleration a_y;
  - vehicle yaw velocity v_ψ;
  - wheel angular velocity for each wheel ω_{ij}.
  - road wheel angle (calculated after measuring the steering wheel angle SWA)

Actually the complexity of the system will be reduced by further assumptions (see following paragraphs).

Moreover, the vehicle parameters can be considered as known, with regard to their nominal values. Possible corrections will be made in the identification phase.

Finally, we assume that non-measured states, such as lateral and longitudinal velocities, are estimated by using a vehicle state observer, over which this work does not focus.

1.8 Further assumptions

1.8.1 Trigonometric approximations

The road wheel angles are "small" (less than $10°$-$15°$):

$$\delta \leq 15^\circ \Longrightarrow \cos\delta \approx 1,\ \sin\ \delta \approx \delta$$

Usually the longitudinal velocity is higher than the lateral velocity and the yaw rate:

$$\begin{cases} v_x \gg v_y \\ v_x \gg dv_\psi \\ v_x \gg l_r v_\psi \\ v_x \gg l_f v_\psi \end{cases} \Longrightarrow \begin{cases} \arctan\left(\frac{v_y+l_f v_\psi}{v_x \mp dv_\psi}\right) = \frac{v_y+l_f v_\psi}{v_x \mp dv_\psi} = \frac{v_y+l_f v_\psi}{v_x} \\ \arctan\left(\frac{v_y-l_r v_\psi}{v_x \mp dv_\psi}\right) = \frac{v_y-l_r v_\psi}{v_x} \\ k_{rj} = \frac{\omega_{rj}R_w-(v_x \mp dv_\psi)}{v_x \mp dv_\psi} = \frac{\omega_{rj}R_w - v_x}{v_x} \end{cases}$$

with the further consequence that slip angles and longitudinal slips can be written in a simpler way:

$$\begin{aligned} \alpha_{fl} &= \alpha_{fr} := \alpha_f \\ \alpha_{rl} &= \alpha_{rr} := \alpha_r \\ k_{rj} &= \tfrac{R_w}{v_x}\omega_{rj} - 1 \end{aligned}$$

1.8.2 Dynamic approximations

Since the road wheel angle is small, we can disregard terms multiplying δ:

$$\begin{aligned} (F_{xfl}+F_{xfr})\sin\delta &\approx (F_{xfl}+F_{xfr})\delta \approx 0 \\ (F_{yfl}+F_{yfr})\sin\delta &\approx (F_{yfl}+F_{yfr})\delta \approx 0 \\ (F_{yfl}-F_{yfr})\sin(\delta)d &\approx (F_{yfl}-F_{yfr})\delta d \approx 0 \end{aligned}$$

Constant tyre vertical loads

We suppose that tyre vertical loads are constant and their distribution, between front and rear axle, depends on the position of the centre of mass of the vehicle; moreover, in each axle, the assumption is that the weight is shared equally between left and right side:

$$\begin{aligned} F_{zfl} &= F_{zfr} = \bar{F}_{zf} = \frac{l_r}{2(l_f + l_r)} mg \\ F_{zrl} &= F_{zrr} = \bar{F}_{zr} = \frac{l_f}{2(l_f + l_r)} mg \end{aligned} \tag{1.9 a}$$

No braking torques

In this paper, brake control is not considered. So the braking torques can be disregarded without loss of generality:

$$T_{Bij} = 0 \quad \text{(i= front, rear; j= left, right)}$$

Real-wheel drive vehicle

In a rear-wheel drive vehicle, the longitudinal slips for the front tyres can be considered substantially equal to zero. As a consequence of this, front longitudinal forces can be neglected:

$$k_{fl} = k_{fr} = 0 \Longrightarrow F_{xfl} = F_{xfr} = 0$$

1.8.3 Kinematic approximations

We assumed the longitudinal velocity as little variable, without quick braking/accelerating actions; in fact we do not focus on a traction control, but on a control based on steering and differential. So we can reduce the number of state variables

$$\dot{v}_x \approx 0 \tag{1.1 a}$$

and the equation (1) becomes a pure algebraic constraint for the model:

$$mv_y v_\psi = F_{xrl} + F_{xrr} \tag{1.1 b}$$

1.8.4 Tyre forces

As a consequence of the previous hypotheses, the expressions of tyre forces are simplified:

$$\begin{aligned} F_{xij} &= F_{xij}(\alpha_{ij}, F_{zij}, k_{ij}, \mu) = F_{xij}(\alpha_i, \bar{F}_{zi}, k_{ij}, 1) = F_{xij}(\alpha_i, k_{ij}) \quad (1.7\text{ a}) \\ &\Longrightarrow \begin{cases} F_{xrl} = F_{xrl}(\alpha_r, k_{rl}) \\ F_{xrr} = F_{xrr}(\alpha_r, k_{rr}) \end{cases} \end{aligned}$$

$$\begin{aligned} F_{yij} &= F_{yij}(\alpha_{ij}, F_{zij}, k_{ij}, \mu) = F_{yij}(\alpha_i, \bar{F}_{zi}, k_{ij}, 1) = F_{yij}(\alpha_i, k_{ij}) \quad (1.8\text{ a}) \\ &\Longrightarrow \begin{cases} F_{yfl} = F_{yfl}(\alpha_f, k_{fl}) = F_{yfl}(\alpha_f, 0) = F_{yfl}(\alpha_f) \\ F_{yfr} = F_{yfr}(\alpha_f, k_{fr}) = F_{yfr}(\alpha_f, 0) = F_{yfr}(\alpha_f) \\ F_{yrl} = F_{yrl}(\alpha_r, k_{rl}) \\ F_{yrr} = F_{yrr}(\alpha_r, k_{rr}) \end{cases} \end{aligned}$$

We define:

$$\begin{aligned} F_{yf}(\alpha_f) &: = F_{yfl}(\alpha_f) + F_{yfr}(\alpha_f) \\ F_{yr}(\alpha_r, k_{rl}) &: = F_{yrl}(\alpha_r, k_{rl}) + F_{yrr}(\alpha_r, k_{rr}) \\ F_{xr}(\alpha_r, \cdot) &= F_{xrl}(\alpha_r, \cdot) = F_{xrl}(\alpha_r, \cdot) \end{aligned}$$

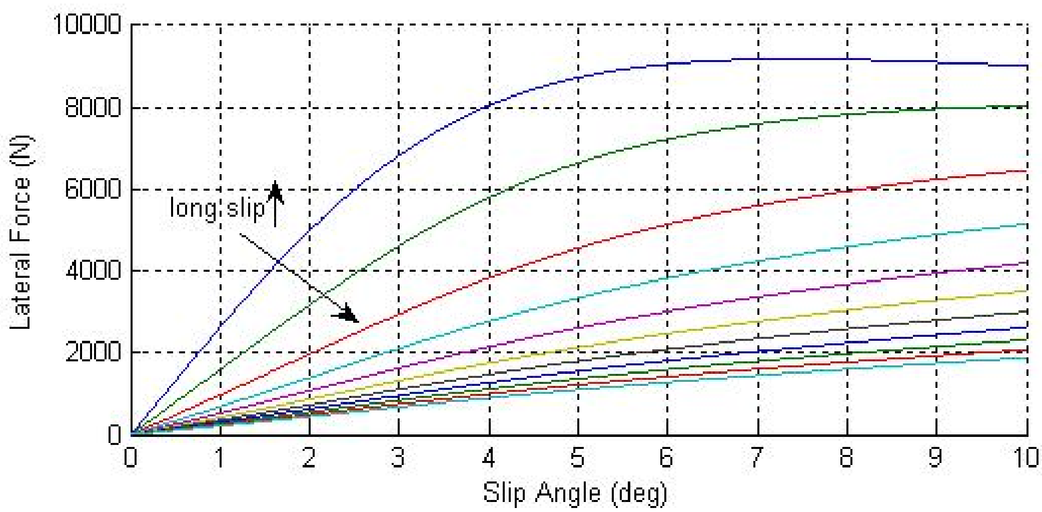
and we make the assumption that rear forces are made of two components:

$$F_{xr}(\alpha_r, k_{rj}) = p_x(\alpha_r)F_{xr}(0, k_{rj}) \tag{1.7 b}$$

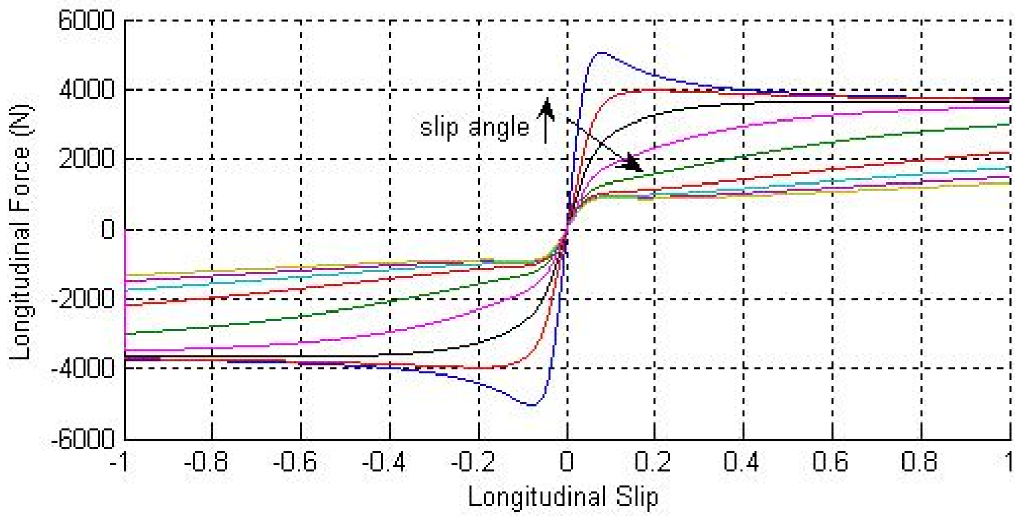
$$F_{yr}(\alpha_r, k_{rj}) = p_y(k_{rj})F_{yr}(\alpha_r, 0) \tag{1.8 b}$$

where $p_x(\alpha_r) \leq 1$ and $p_y(k_{rj}) \leq 1$ are "penalty functions" (see Chapter 2 for details).

This assumption is likely to be valid, since the tyre lateral force usually decreases in condition of combined (longitudinal and lateral) slip, if compared to the situation of pure lateral slip. The same behavior is exhibited by the longitudinal force. These plots, obtained through simulations on a real vehicle, furtherly justify the assumption



Tyre Lateral Force



Tyre Longitudinal Force

1.8.5 Active Differential (AD)

The Active Differential adjusts the distribution of torque in the rear axle:

$$
\begin{aligned}
T_{Trl} &= T_{roll} - T_{AD} \\
T_{Trr} &= T_{roll} + T_{AD}
\end{aligned}
$$

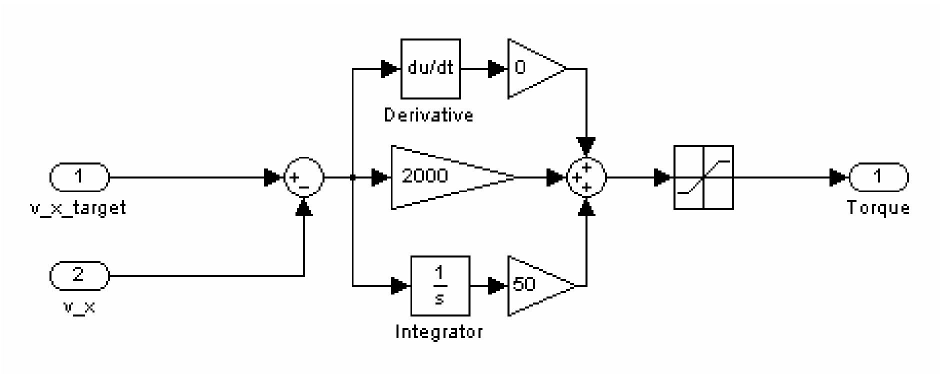
where T_{roll} is distributed to the wheels when AD is not active.

Since the longitudinal velocity is little variable, we assumed $\dot{v}_x \approx 0$ and so we can consider

$$\begin{cases} T_{roll} \approx 0 \\ T_{Trl} = -T_{AD} \\ T_{Trr} = T_{AD} \end{cases} \implies \begin{cases} J_w \ \dot{\omega}_{rl} = T_{rl} - R_w F_{xrl} = -T_{AD} - R_w F_{xrl} \\ J_w \ \dot{\omega}_{rr} = T_{rr} - R_w F_{xrr} = T_{AD} - R_w F_{xrr} \end{cases} \tag{1.11 a}$$

$$\begin{aligned} \dot{k}_{rl} &= \frac{R_w}{v_x}\dot{\omega}_{rl} = \frac{R_w}{J_w v_x}(-T_{AD} - R_w F_{xrl}) = -\frac{R_w^2}{J_w v_x}F_{xrl} - \frac{R_w}{J_w v_x}T_{AD} = \\ &= -\frac{R_w^2}{J_w v_x}p_x(\alpha_r)F_{xr}(0, k_{rl}) - \frac{R_w}{J_w v_x}T_{AD} \\ \dot{k}_{rr} &= \frac{R_w}{v_x}\dot{\omega}_{rr} = \frac{R_w}{J_w v_x}(T_{AD} - R_w F_{xrr}) = -\frac{R_w^2}{J_w v_x}F_{xrr} + \frac{R_w}{J_w v_x}T_{AD} = \\ &= -\frac{R_w^2}{J_w v_x}p_x(\alpha_r)F_{xr}(0, k_{rr}) + \frac{R_w}{J_w v_x}T_{AD} \end{aligned} \tag{1.16}$$

Actually, $T_{roll} \neq 0$ even if $\dot{v}_x = 0$, since the engine has to provide torque to keep the longitudinal velocity constant, because of the action of non-conservative forces, like friction and air resistance. This torque is calculated, in real vehicles, through a velocity control loop like this



Speed Controller

Since $F_{xr}(0,k) = -F_{xr}(0,-k)$ (see the following chapter), with the further hypothesis:

$$k_{rl}(t_0) = k_{rr}(t_0) = 0 \Longrightarrow F_{xrl}(t_0) = F_{xrr}(t_0) = 0 \Longrightarrow \dot{k}_{rl}(t) = -\dot{k}_{rr}(t) \forall t$$

and so

$$k_{rl}(t) = -k_{rr}(t).$$

1.9 State-space vehicle model

The congruence equations have assumed a simplified form:

$$\alpha_f = \delta - \frac{v_y + l_f v_\psi}{v_x} \tag{1.4 a}$$

$$\alpha_r = -\frac{v_y - l_r v_\psi}{v_x} \tag{1.5 a}$$

After the previous assumptions, the plant complexity was mitigated, so that we can define a space-state model with only 3 state variables: v_y, v_ψ, k_{rr}.

The dynamic equations are:

$$m(\dot{v}_y + v_x v_\psi) = F_{yf} + F_{yr} \tag{1.2 a}$$

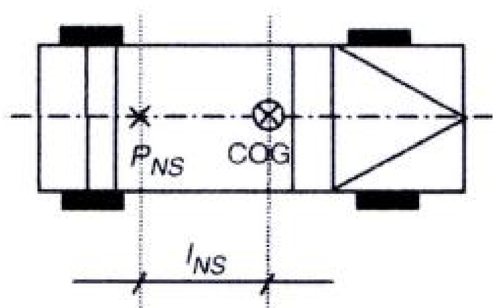
$$J_z v_\psi = F_{yf} l_f - F_{yr} l_r - (F_{xrl} - F_{xrr})d = F_{yf} l_f - F_{yr} l_r + 2F_{xrr} d \tag{1.3 a}$$

$$\dot{k}_{rr} = -\frac{R_w^2}{J_w v_x} F_{xrr} + \frac{R_w}{J_w v_x} T_{AD} \tag{1.16 a}$$

Now we will consider the following transformation (*Neutral Steering Point*):

$$v_{yNS} = v_y - l_{NS} v_\psi \tag{1.17}$$

$$l_{NS} = \frac{J_z}{m l_f}$$



Definition of Neutral Steering Point

and so:

$$\begin{aligned}
\dot{v}_{yNS} &= \dot{v}_y - l_{NS}\dot{v}_\psi = -v_x v_\psi + \frac{F_{yf}+F_{yr}}{m} - l_{NS}\frac{F_{yf}l_f - F_{yr}l_r + 2F_{xrr}d}{J_z} \qquad (1.2\ \text{b}) \\
&= -v_x v_\psi + \frac{F_{yf}+F_{yr}}{m} - \frac{J_z}{ml_f}\frac{F_{yf}l_f - F_{yr}l_r + 2F_{xrr}d}{J_z} = \\
&= -v_x v_\psi + \frac{F_{yf}+F_{yr}}{m} - \frac{F_{yf}l_f - F_{yr}l_r + 2F_{xrr}d}{ml_f} = \\
&= -v_x v_\psi + \frac{F_{yf}+F_{yr}}{m} - \left(\frac{F_{yf}l_f}{ml_f} - \frac{F_{yr}l_r}{ml_f} + \frac{2F_{xrr}d}{ml_f}\right) = \\
&= -v_x v_\psi + \frac{F_{yr}}{m} + \frac{F_{yr}l_r}{ml_f} - \frac{2F_{xrr}d}{ml_f} = \\
&= -v_x v_\psi + \frac{F_{yr}\left(l_f + l_r\right) - 2F_{xrr}d}{ml_f}
\end{aligned}$$

The newly introduced constant l_{NS} fixes the neutral steering point on the vehicle longitudinal axis whose lateral velocity variation, according to equation 1.2 b, is independent from the front lateral force. This point is on the back of the vehicle centre of mass, at a distance of exactly l_{NS} (eq. 1.17). In typical cases, P_{ns} is very close to the rear axle.

From the control point of view, the consequence is that lateral velocity tracking is influenced only by the AD input and independent from the AFS. That will be shown in simulations.

We define:

$$x = \begin{bmatrix} v_{yNS} \\ v_\psi \\ k_r \end{bmatrix}$$

$$u = \begin{bmatrix} \delta \\ T_{AD} \end{bmatrix}$$

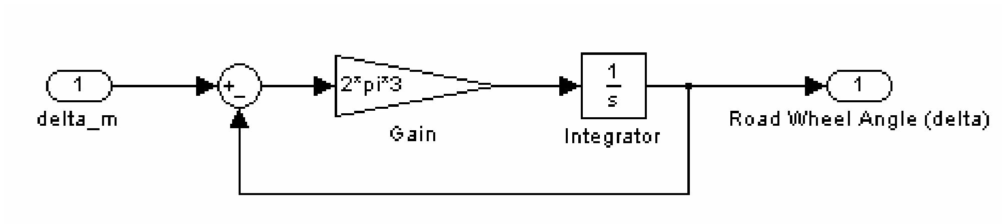
$$y = \begin{bmatrix} v_{yNS} \\ v_\psi \end{bmatrix}$$

The dynamic equations are the following:

$$
\begin{aligned}
\dot{x} &= \begin{bmatrix} \dot{v}_{yNS} \\ \dot{v}_{\psi} \\ \dot{k}_r \end{bmatrix} = \\
&= \begin{bmatrix} -v_x v_\psi + \frac{\left(l_f + l_r\right) F_{yr}(\alpha_r, k_r) - 2d F_{xrr}(\alpha_r, k_r)}{m l_f} \\ \frac{F_{yf}(\alpha_f) l_f - F_{yr}(\alpha_r, k_r) l_r + 2 F_{xrr}(\alpha_r, k_r) d}{J_z} \\ -\frac{R_w^2}{J_w v_x} F_{xrr}(\alpha_r, k_r) + \frac{R_w}{J_w v_x} T_{AD} \end{bmatrix} = \\
&= \begin{bmatrix} -v_x v_\psi + \frac{p_y(k_r) F_{yr}(\alpha_r, 0) l - p_x(\alpha_r) F_{xrr}(0, k_r) t}{m l_f} \\ \frac{F_{yf}(\alpha_f) l_f - p_y(k_r) F_{yr}(\alpha_r, 0) l_r + p_x(\alpha_r) F_{xrr}(0, k_r) t}{J_z} \\ -\frac{R_w^2}{J_w v_x} p_x(\alpha_r) F_{xrr}(0, k_r) + \frac{R_w}{J_w v_x} T_{AD} \end{bmatrix} = \\
&= \begin{bmatrix} -v_x v_\psi + \frac{p_y(k_r) F_{yr}(-\frac{v_y - l_r v_\psi}{v_x}) l - p_x(-\frac{v_y - l_r v_\psi}{v_x}) F_{xr}(k_r) t}{m l_f} \\ \frac{F_{yf}(\delta - \frac{v_y + l_f v_\psi}{v_x}) l_f - p_y(k_r) F_{yr}(-\frac{v_y - l_r v_\psi}{v_x}) l_r + p_x(-\frac{v_y - l_r v_\psi}{v_x}) F_{xr}(k_r) t}{J_z} \\ -\frac{R_w^2}{J_w v_x} p_x(-\frac{v_y - l_r v_\psi}{v_x}) F_{xr}(k_r) + \frac{R_w}{J_w v_x} T_{AD} \end{bmatrix} = \\
&= \begin{bmatrix} -v_x v_\psi + \frac{p_y(k_r) F_{yr}(-\frac{v_{yNS} - (l_r - l_{NS}) v_\psi}{v_x}) l - p_x(-\frac{v_{yNS} - (l_r - l_{NS}) v_\psi}{v_x}) F_{xr}(k_r) t}{m l_f} \\ \frac{F_{yf}(\delta - \frac{v_{yNS} + (l_f + l_{NS}) v_\psi}{v_x}) l_f - p_y(k_r) F_{yr}(-\frac{v_{yNS} - (l_r - l_{NS}) v_\psi}{v_x}) l_r + p_x(-\frac{v_{yNS} - (l_r - l_{NS}) v_\psi}{v_x}) F_{xr}(k_r) t}{J_z} \\ -\frac{R_w^2}{J_w v_x} p_x(-\frac{v_{yNS} - (l_r - l_{NS}) v_\psi}{v_x}) F_{xr}(k_r) + \frac{R_w}{J_w v_x} T_{AD} \end{bmatrix} = \\
&= \begin{bmatrix} -v_x x_2 + \frac{p_y(x_3) F_{yr}(-\frac{x_1 - (l_r - l_{NS}) x_2}{v_x}) l - p_x(-\frac{x_1 - (l_r - l_{NS}) x_2}{v_x}) F_{xr}(x_3) t}{m l_f} \\ \frac{F_{yf}(u_1 - \frac{x_1 + (l_f + l_{NS}) x_2}{v_x}) l_f - p_y(x_3) F_{yr}(-\frac{x_1 - (l_r - l_{NS}) x_2}{v_x}) l_r + p_x(-\frac{x_1 - (l_r - l_{NS}) x_2}{v_x}) F_{xr}(x_3) t}{J_z} \\ -\frac{R_w^2}{J_w v_x} p_x(-\frac{x_1 - (l_r - l_{NS}) x_2}{v_x}) F_{xr}(x_3) + \frac{R_w}{J_w v_x} u_2 \end{bmatrix}
\end{aligned}
$$

In the vehicle, the wheel angle δ is governed by the following dynamic equation:

$$\dot{\delta} = -\frac{1}{\tau_M} \delta + \frac{1}{\tau_M} \delta_M \tag{1.18}$$



Filter (Steering System)

So we can define a new state variable $x_4 = \delta$ so that F_{yf} only depends on the state variables; moreover we substitute the input u_1 with δ_M.

Finally, the plant has this structure:

$$\dot{x} = f(x) + G\ u$$

where:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} v_{yNS} \\ v_\psi \\ k_r \\ \delta \end{bmatrix}$$

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \delta_M \\ T_{AD} \end{bmatrix}$$

$$f(x) = \begin{bmatrix} -v_x x_2 + \frac{p_y(x_3) F_{yr}(-\frac{x_1-(l_r-l_{NS})x_2}{v_x}) l - p_x(-\frac{x_1-(l_r-l_{NS})x_2}{v_x}) F_{xr}(x_3) t}{m l_f} \\ \frac{F_{yf}(x_4-\frac{x_1+(l_f+l_{NS})x_2}{v_x}) l_f - p_y(x_3) F_{yr}(-\frac{x_1-(l_r-l_{NS})x_2}{v_x}) l_r + p_x(-\frac{x_1-(l_r-l_{NS})x_2}{v_x}) F_{xr}(x_3) t}{J_z} \\ -\frac{R_w^2}{J_w v_x} p_x(-\frac{x_1-(l_r-l_{NS})x_2}{v_x}) F_{xr}(x_3) \\ -\frac{1}{\tau_M} x_4 \end{bmatrix}$$

$$G = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \frac{R_w}{J_w v_x} \\ \frac{1}{\tau_M} & 0 \end{bmatrix}$$

Chapter 2

Identification

2.1 Vehicle mass identification

The mass value can be fitted by 2 equations:

$$m = \frac{F_{yf} + F_{yr}}{a_y}$$

$$m = \frac{F_{zfl} + F_{zfr} + F_{zrl} + F_{zrr}}{g}$$

In several simulations, and different manoeuvres, we find that the value of the vehicle mass is about

$$1877\ kg$$

The nominal value is 1700 kg, so the correction is about 10.43%. The consequence is that the longitudinal, lateral and vertical forces, in the plant, will be higher than expected.

2.2 Other parameters

2.2.1 Vehicle lengths

The front and rear vehicle lengths, l_f and l_r, would be functions of time, because the position of the centre of mass changes during the manoeuvres.

Since we have to decide a constant value for these lengths, we consider the nominal values

$$\begin{aligned} l_f &= 1.5285\ m \\ l_r &= 1.3782\ m \end{aligned}$$

as good values for the plant parameters. They correspond to the position of the centre of mass during a cornering manoeuvre with steering angle of about 25.2°; without steering, instead, the centre of mass moves forward of 1.21 cm, so this condition corresponds to the maximum value of l_r and the minimum value of l_f.

2.2.2 Wheel parameters

It is decided to keep the nominal values of parameters for the wheel ones, as well, because no particular assumptions have been made on the wheels equations. Their behavior in the plant model is the same if compared to the real vehicle model, therefore the original parameters fit well

$$\begin{aligned} R_w &= 0.329\ m \\ J_w &= 1\ kg\ m^2 \end{aligned}$$

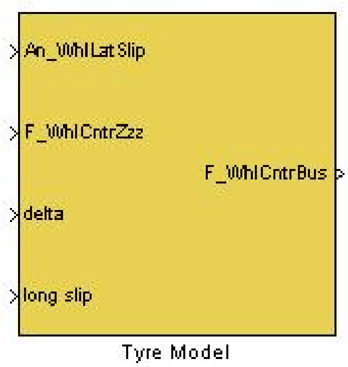
as confirmed by experiments on the real plant and on the model.

2.3 Tyre characteristics identification

We identificated the following functions:

$$
\begin{aligned}
&F_{yf}(\alpha_f) \\
&F_{yr}(\alpha_r) \qquad \text{(no longitudinal slip)} \\
&F_{xr}(k_r) \qquad \text{(no lateral slip)} \\
&p_y(k_r) \\
&p_x(\alpha_r)
\end{aligned}
$$

by several simulations on the real vehicle and on a tyre model, whose behavior is the same of the vehicle tyres.



Tyre Model

2.3.1 Experimental tests

We needed some test manoeuvres on the vehicle to get and validate some characteristics to insert in the model. They have been carried out in steady-state conditions. It means that, after a constant (steering or differential) input, the equilibrium situation is not achieved until some time has elapsed after the system is started. The consequence of such kind of identification is that, for each different applied input, the plant and the real vehicle will have the same equilibrium point.

We describe the following manoeuvres:

1. Constant SWA, without torque input

2. Constant (traction) torque, without steering input

3. Constant (differential) torque, constant SWA

Constant SWA, without torque input

In the first manoeuvres set, we apply a constant SWA input, without torque input, to the real vehicle. The longitudinal velocity is kept constant to a value v_x. The simulation is repeated for several values of SWA, until stability is saved. Since no differential input is applied, according to the vehicle plant equations, we have

$$k_r = 0$$

So, the goal of the first set of simulations is the definition of the lateral characteristics, for both of the axles, in condition of pure lateral slip (no longitudinal slip). The equilibrium equations are:

$$\begin{aligned} \dot{v}_y &= 0 \\ \dot{v}_\psi &= 0 \end{aligned}$$

In detail:

$$\begin{aligned}
\dot{v}_y &= -v_x v_\psi + \frac{F_{yf}(\alpha_f) + p_y(0)F_{yr}(\alpha_r, 0)}{m} = \\
&= -v_x v_\psi + \frac{F_{yf}(\alpha_f) + F_{yr}(\alpha_r, 0)}{m} = 0 \\
\dot{v}_\psi &= \frac{F_{yf}(\alpha_f)l_f - p_y(0)F_{yr}(\alpha_r, 0)l_r + p_x(\alpha_r)F_{xrr}(0, k_r)t}{J_z} = \\
&= \frac{F_{yf}(\alpha_f)l_f - F_{yr}(\alpha_r, 0)l_r}{J_z} = 0
\end{aligned}$$

and so

$$\begin{aligned}
F_{yf,id}(\alpha_f) + F_{yr,id}(\alpha_r, 0) &= m v_x v_{\psi,mis} \\
F_{yf,id}(\alpha_f)l_f &= F_{yr,id}(\alpha_r, 0)l_r
\end{aligned}$$

Finally, this procedure yields the following results:

$$\begin{aligned}
F_{yf,id}(\alpha_f) &= \frac{l_r}{l_f + l_r} m v_x v_{\psi,mis} \\
F_{yr,id}(\alpha_r, 0) &= \frac{l_f}{l_f + l_r} m v_x v_{\psi,mis}
\end{aligned}$$

For each different simulation, these steps were followed:

1. Measurements of

   - vehicle lateral velocity
   - vehicle yaw rate

   detected by the vehicle sensors.

2. Calculation of slip angles by congruence equations:

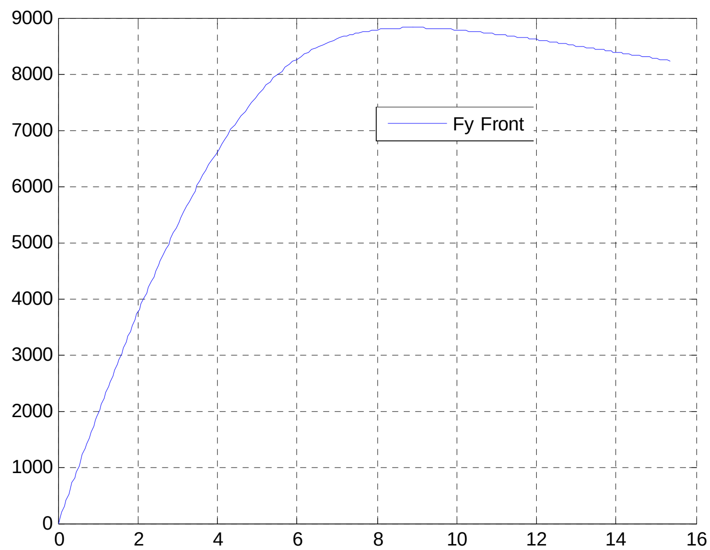
$$\alpha_f = \delta - \frac{v_y + l_f v_\psi}{v_x} \tag{1.4 a}$$

$$\alpha_r = -\frac{v_y - l_r v_\psi}{v_x} \qquad (1.5\ a)$$

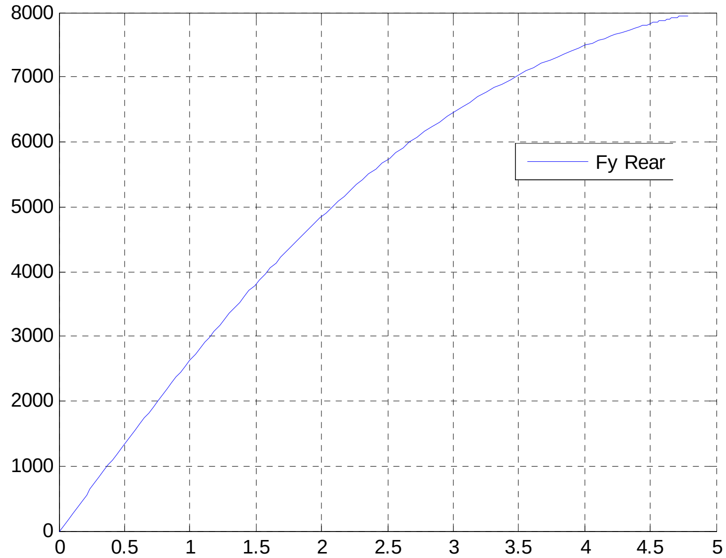
3. Calculation of tyre lateral forces by the following equilibrium expressions:

$$\begin{aligned} F_{yf,id}(\alpha_f) &= \frac{l_r}{l_f + l_r} m v_x v_{\psi,mis} \\ F_{yr,id}(\alpha_r, 0) &= \frac{l_f}{l_f + l_r} m v_x v_{\psi,mis} \end{aligned}$$

The identified characteristics are the following ones



Front Lateral Characteristic



Rear Lateral Characteristic

It can be seen that there are some steady states corresponding to the decreasing part of the front lateral characteristic. On the contrary, only points belonging to the increasing part of the rear lateral characteristic can be identified, because beyond the saturation point the vehicle dynamics goes unstable.

Constant (traction) torque, without steering input

In the second manouvres set, we apply a constant traction torque input T_T, without steering, to the real vehicle. The simulation is repeated for several values of torque, until stability, for the rear longitudinal characteristic, is saved. Since no steering is applied, according to the vehicle plant equations, if the initial conditions are zero, we have

$$\begin{aligned}
\delta &= 0 \\
v_y &= 0 \\
v_\psi &= 0 \\
\alpha_r &= 0
\end{aligned}$$

So, the goal is the definition of the longitudinal characteristic, for the rear axle, in condition of pure longitudinal slip (no lateral slip). The equilibrium equation is:

$$\dot{k}_r = 0$$

In detail:

$$\begin{aligned}
\dot{k}_r &= -\frac{R_w^2}{J_w v_x} p_x(0) F_{xrr}(0, k_r) + \frac{R_w}{J_w v_x} T_T = \\
&= -\frac{R_w^2}{J_w v_x} F_{xrr}(0, k_r) + \frac{R_w}{J_w v_x} T_T = 0
\end{aligned}$$

and so

$$F_{xrr,id}(0, k_r) = \frac{T_T}{R_w}$$

So, for each different simulation, these steps were followed:

1. Measurement of

   - rear right wheel angular velocity
   - vehicle longitudinal velocity

   detected by the vehicle sensors

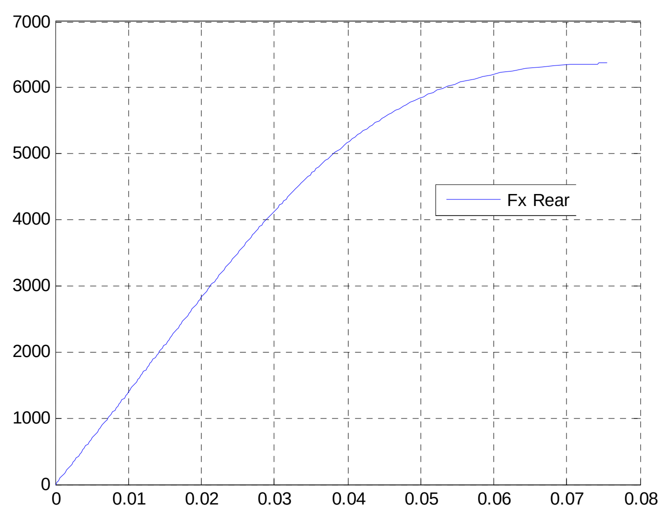
2. Calculation of the rear right longitudinal slip by the slip equation:

$$k_{rr} = \frac{\omega_{rr} R_w - v_x}{v_x}$$

3. Calculation of rear right tyre longitudinal force by the following equilibrium expression:

$$F_{xrr,id}(0, k_r) = \frac{T_T}{R_w}$$

The identified characteristic is the following one



Rear Longitudinal Characteristic

Constant (differential) torque, constant SWA

Finally, in the last manoeuvres set, we apply both a constant SWA input and a differential torque input T_{AD}, to the real vehicle. The longitudinal velocity can be considered as constant to a value v_x, because of the absence of the traction torque. The simulation is repeated for several values of SWA and T_{AD}, until stability is saved.

The goal of the last part of the identification procedure is the definition of weight functions, which penalize the tyre forces in conditions of combined lateral-longitudinal slip.

The equilibrium equations are:

$$\begin{aligned}\dot{v}_y &= 0\\ \dot{v}_\psi &= 0\\ \dot{k}_r &= 0\end{aligned}$$

In detail:

$$\begin{aligned}\dot{v}_y &= -v_x v_\psi + \frac{F_{yf}(\alpha_f) + p_y(k_r)F_{yr}(\alpha_r,0)}{m} = 0\\ \dot{v}_\psi &= \frac{F_{yf}(\alpha_f)l_f - p_y(k_r)F_{yr}(\alpha_r,0)l_r + p_x(\alpha_r)F_{xrr}(0,k_r)t}{J_z} = 0\end{aligned}$$

and so:

$$\begin{aligned}F_{yf,id}(\alpha_f) + p_y(k_r)F_{yr,id}(\alpha_r,0) &= m v_x v_{\psi,mis}\\ p_y(k_r)F_{yr,id}(\alpha_r,0)l_r - p_x(\alpha_r)F_{xrr,id}(0,k_r)t &= F_{yf,id}(\alpha_f)l_f\end{aligned}$$

Finally, this procedure yields the following results:

$$p_{y,id}(k_r) = \frac{m v_x v_{\psi,mis} - F_{yf,id}(\alpha_f)}{F_{yr,id}(\alpha_r,0)}$$

$$p_{x,id}(\alpha_r) = \frac{F_{yf,id}(\alpha_f)l_f - p_{y,id}(k_r)F_{yr,id}(\alpha_r,0)l_r}{F_{xrr,id}(0,k_r)t}$$

For each different simulation, these steps were followed:

1. Measurements of

   - lateral velocity
   - yaw rate
   - rear right wheel angular velocity

detected by the vehicle sensors.

2. Calculation of slip angles and rear right longitudinal slip by the following equations:

$$\alpha_f = \delta - \frac{v_y + l_f v_\psi}{v_x} \tag{1.4 a}$$

$$\alpha_r = -\frac{v_y - l_r v_\psi}{v_x} \tag{1.5 a}$$

$$k_{rr} = \frac{\omega_{rr} R_w - v_x}{v_x}$$

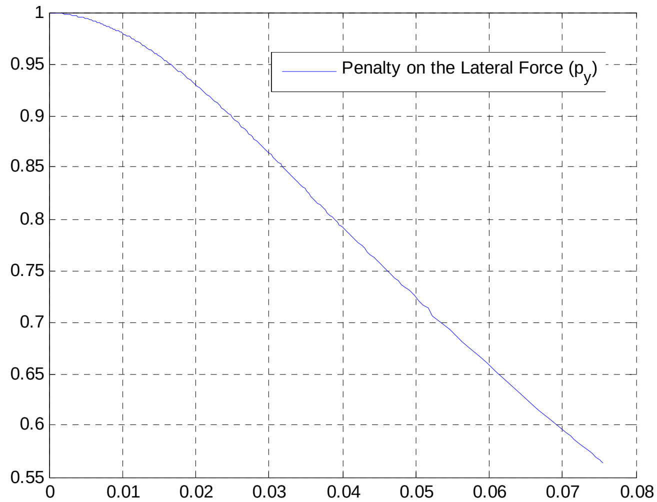
3. Calculation of the weight functions by the following equilibrium expressions:

$$p_{y,id}(k_r) = \frac{m v_x v_{\psi,mis} - F_{yf,id}(\alpha_f)}{F_{yr,id}(\alpha_r, 0)}$$

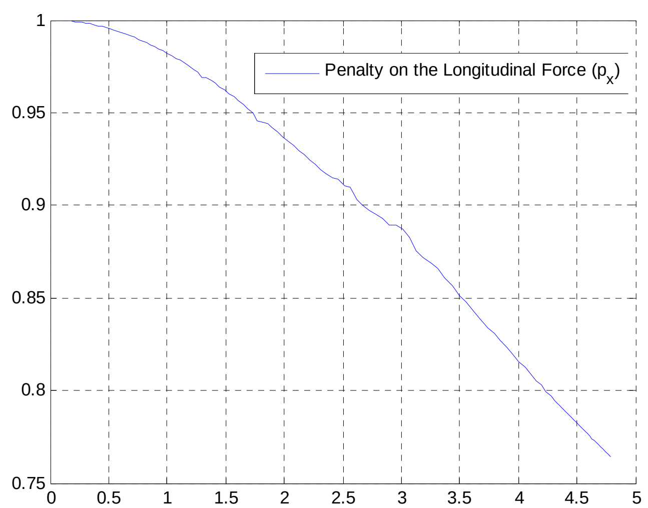
$$p_{x,id}(\alpha_r) = \frac{F_{yf,id}(\alpha_f) l_f - p_{y,id}(k_r) F_{yr,id}(\alpha_r, 0) l_r}{F_{xrr,id}(0, k_r) t}$$

where $F_{yf,id}$, $F_{yf,id}$, $F_{yf,id}$ are the characteristics that were identified in the previous sets of manoeuvres, with pure lateral or longitudinal slip conditions.

The identified weight functions are the following ones



Lateral Weight Function



Longitudinal Weight Function

2.3.2 Fitting Magic Formulas for tyre characteristics

By the experimental data, we can interpolate some curves. These functions are not ideal, because they are affected by noises of sensors and other errors.

Moreover, we need that the tyre forces respect further constraints, depending of the tyre model:

- Longitudinal and lateral forces "odd functions"

$$\begin{aligned} F_{yf}(-\alpha_f) &= -F_{yf}(\alpha_f) \\ F_{yr}(-\alpha_r, 0) &= -F_{yr}(\alpha_r, 0) \\ F_{xr}(0, -k_r) &= -F_{xr}(0, k_r) \end{aligned}$$

- Longitudinal and lateral forces monotonically increasing till the saturation point

$$\begin{aligned} \frac{dF_{yf}}{d\alpha_f} &> 0 & \alpha_f \in [-\alpha_{f,sat}\ , \alpha_{f,sat}] \\ \frac{\partial F_{yr}}{\partial \alpha_r} &> 0 & \alpha_r \in [-\alpha_{r,sat}\ , \alpha_{r,sat}] \\ \frac{\partial F_{xr}}{\partial k_r} &> 0 & k_r \in [-k_{r,sat}\ , k_{r,sat}] \end{aligned}$$

- Derivative of such functions (*relative stiffnesses*) maximum in the origin point

$$\begin{aligned} \frac{d^2 F_{yf}}{d\alpha_f^2} &= 0 & \alpha_f = 0 \\ \frac{\partial^2 F_{yr}}{\partial \alpha_r^2} &= 0 & \alpha_r = 0 \\ \frac{\partial^2 F_{xr}}{\partial k_r^2} &= 0 & k_r = 0 \end{aligned}$$

then monotonically decreasing for positive values of x

$$\begin{aligned}
\frac{d^2 F_{yf}}{d\alpha_f^2} &< 0 & \alpha_f > 0 \\
\frac{\partial^2 F_{yr}}{\partial\alpha_r^2} &< 0 & \alpha_r > 0 \\
\frac{\partial^2 F_{xr}}{\partial k_r^2} &< 0 & k_r > 0
\end{aligned}$$

Such constraints can be easily respected by using particular functions called Pacejka's Magic Formulas. The official Pacejka formula goes like this:

$$y(x) = D \sin \left\{ C \arctan \left[Bx - E \left(Bx - \arctan \left(Bx \right) \right) \right] \right\}$$

where

- B is the stiffness factor
- C is the shape factor
- D is the peak value of the curve
- E is the curvature factor

For the sake of simplicity, we will consider $E = 0$.

These curves, as written before, satisfy some good features. So, we decided to find the Magic curve that best approximate the $F_{yf,id}(\alpha_f)$, $F_{yr,id}(\alpha_r, 0)$ and $F_{xr,id}(0, k_r)$, described in the previous subsection.

The problem of the determination of the coefficients B, C, D which best fit the data samples is a problem of non-linear approximation, so it is not trivial to solve. However, by analyzing the identified curves, we can argue the maxima of these curves, usually near to the last identified point of the set, and the corresponding peak values. So, if x_m is the maximum and $F(x_m)$

the peak value, we can approximately choose

$$\begin{aligned} D &= F(x_m) \\ C &\in [C_{\min}, C_{\max}] \\ B &= \frac{\tan\left(\frac{\pi}{2C}\right)}{x_m} \end{aligned}$$

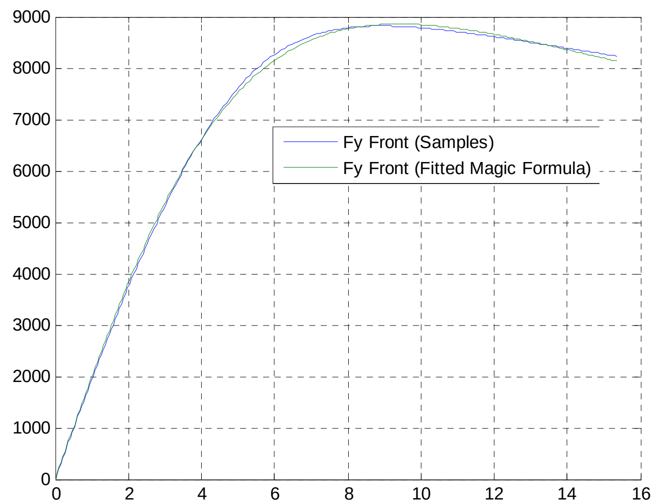
and have good values for a Magic Formula, fitting a sample function $F(x)$. The best values for B, C, D can be obtained letting the parameter C span in an interval $[C_{\min}, C_{\max}]$, then calculating the corresponding value for B.

The fitted Magic Formulas, for the identified characteristics, correspond to the following values of parameters

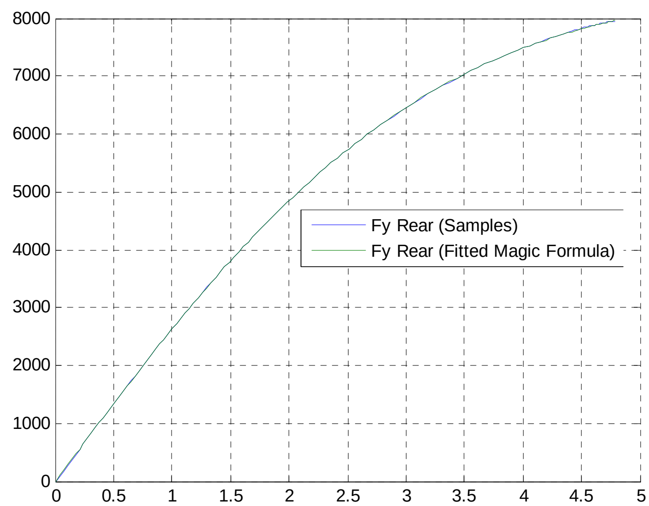
$$\begin{aligned} B_{yf} &= 7.2 \\ C_{yf} &= 1.81 \\ D_{yf} &= 8854 \end{aligned}$$

$$\begin{aligned} B_{yr} &= 11 \\ C_{yr} &= 1.68 \\ D_{yr} &= 8394 \end{aligned}$$

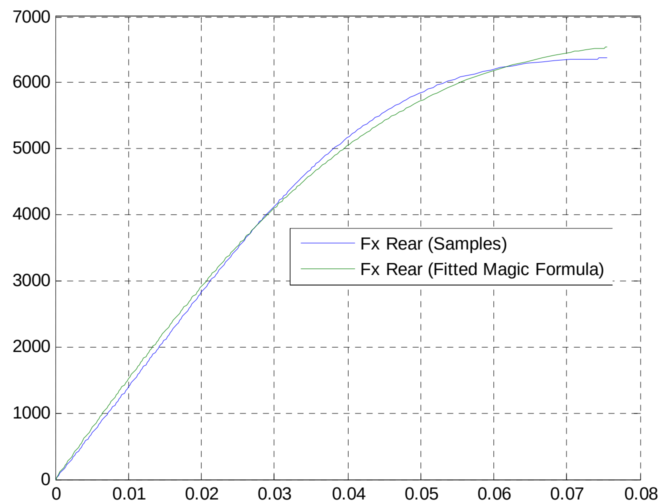
$$\begin{aligned} B_{xr} &= 11.77 \\ C_{xr} &= 1.98 \\ D_{xr} &= 6590 \end{aligned}$$



Fitted Magic Formula for Fy Front



Fitted Magic Formula for Fy Rear



Fitted Magic Formula for Fx Rear

2.3.3 Normalized Tyre Characteristics

The tyre characteristics were built through the following equations

$$\begin{aligned} F_{yf,id}(\alpha_f) &= \frac{l_r}{l_f + l_r} m v_x v_{\psi,mis} \\ F_{yr,id}(\alpha_r, 0) &= \frac{l_f}{l_f + l_r} m v_x v_{\psi,mis} \end{aligned}$$

$$F_{xrr,id}(0, k_r) = \frac{T_T}{R_w}$$

where l_f, l_r, m, R_w are parameters that were previously identified. So these curves depend on these values. In presence of a change of these parameters, there is no need to repeat the experiments again. To get characteristics that are independent from the parameters, this curves can be normalized:

$$\begin{aligned}
F_{yf,norm}(\alpha_f) &= \frac{l_f + l_r}{ml_r} F_{yf,id}(\alpha_f) \\
F_{yr,norm}(\alpha_r, 0) &= \frac{l_f + l_r}{ml_r} F_{yr,id}(\alpha_r, 0) \\
F_{xrr,norm}(0, k_r) &= \frac{F_{xrr,id}(0, k_r)}{R_w}
\end{aligned}$$

The corresponding fitted magic formulas can be normalized as well, in the same way.

2.3.4 Polynomial approximations for weight functions

There are no "Magic Formulas" available to approximate experimental values of weight functions. However, we expect that such curves have some basic properties:

- $p_{x,id}(\alpha_r)$ and $p_{y,id}(k_r)$ bounded in $[0, 1]$
- $p_{x,id}(\alpha_r)$ and $p_{y,id}(k_r)$ are "even" functions

$$\begin{aligned}
p_{x,id}(-\alpha_r) &= p_{x,id}(\alpha_r) \\
p_{y,id}(-k_r) &= p_{y,id}(k_r)
\end{aligned}$$

- $p_{x,id}(\alpha_r)$ and $p_{y,id}(k_r)$ are maximum in the origin point

$$\begin{aligned}
p_{x,id}(0) &= 1 \\
p_{y,id}(0) &= 1 \\
\frac{dp_{x,id}}{d\alpha_r} &| \quad_{\alpha_r=0} = 0 \\
\frac{dp_{y,id}}{dk_r} &| \quad_{k_r=0} = 0
\end{aligned}$$

and

$$
\begin{aligned}
p_{x,id}(\alpha_r) &< 1 \qquad \alpha_r \neq 0 \quad \text{(penalty function)} \\
p_{y,id}(k_r) &< 1 \qquad k_r \neq 0 \quad \text{(penalty function)}
\end{aligned}
$$

We can correct single points where the previous conditions are not verified. Another possibility is to approximate the identified weight functions by a polynomial

$$f(x) = 1 + a_1 x^2 + a_2 x^4 + ... + a_n x^{2n} \qquad n \in \mathbb{N}$$

This function has the following features:

$$
\begin{aligned}
f(x) &= 1 \\
f'(0) &= 0 \\
f(-x) &= f(x) \qquad \text{(even function)}
\end{aligned}
$$

For a good choice of the coefficients a_i, this function can be a "penalty function" ($f(x) < 1$ if $x \in [-x_m, x_m]$, where x_m is an opportune value) and can be a good approximation of the identified weight function.

Differently from the problem of optimization for the parameters of the Magic Formulas, the present problem can be written as a linear optimization problem, and so it can be solved by the Least Square Method (see Appendix A).

Increasing the order $2n$ of the approximating polynomial, $n \in \mathbb{N}$, we get a better and better fitting. Actually, if $n > 5$, the fitting does not get better anymore. We got the best fitting of the identified weight curves through these functions

$$
\begin{aligned}
p_{x,fit}(\alpha_r) &= 1 + c_{x1}x^2 + c_{x2}x^4 + c_{x3}x^6 + c_{x4}x^8 + c_{x5}x^{10} & \alpha_r \in [-\alpha_{r,sat}, \alpha_{r,sat}] \\
p_{y,fit}(k_r) &= 1 + c_{y1}x^2 + c_{y2}x^4 + c_{y3}x^6 + c_{y4}x^8 + c_{y5}x^{10} & k_r \in [-k_{r,sat}, k_{r,sat}]
\end{aligned}
$$

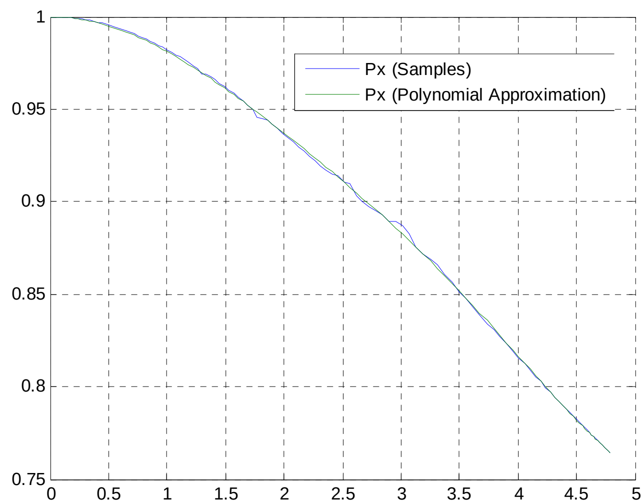
with

$$
\begin{aligned}
\alpha_{r,sat} &= 0.0835\ (rad) & \text{(identified point)} \\
k_{r,sat} &= 0.0756 & \text{(identified point)}
\end{aligned}
$$

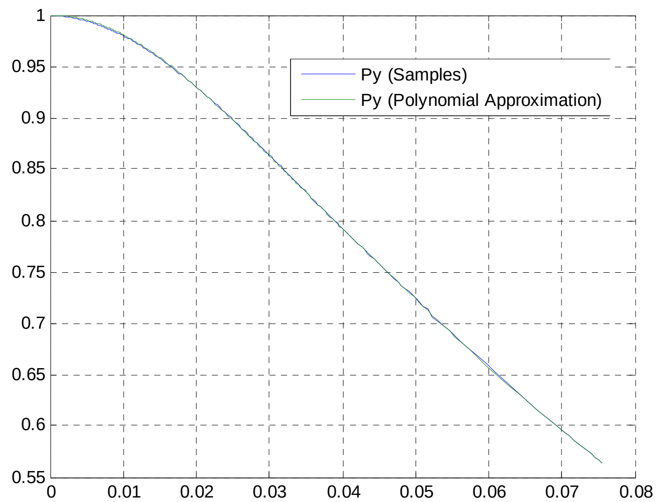
and where the coefficients assume the following values

$$
\begin{aligned}
c_{x1} &= -66.63 \\
c_{x2} &= 16267.29 \\
c_{x3} &= -3775683 \\
c_{x4} &= 425399162 \\
c_{x5} &= -1.74 * 10^{10}
\end{aligned}
$$

$$
\begin{aligned}
c_{y1} &= -197.37 \\
c_{y2} &= 62528.08 \\
c_{y3} &= -15906291 \\
c_{y4} &= 2.23 * 10^{9} \\
c_{y5} &= -1.25 * 10^{11}
\end{aligned}
$$



Polynomial approximation for the Longitudinal Weight Function



Polynomial approximation for the Lateral Weight Function

2.4 Conclusions

The approach that has been used in this identification phase is quite simple. In particular, the problem of identificating 2 functions of 2 variables, $F_{yr}(\alpha_r, k_r)$ and $F_{xr}(\alpha_r, k_r)$, was reduced in complexity, because each of these functions has been considered the product of two scalar functions

$$F_{xr}(\alpha_r, k_r) = p_x(\alpha_r) F_{xr}(0, k_r) \qquad (1.7\text{ b})$$

$$F_{yr}(\alpha_r, k_r) = p_y(k_{rj}) F_{yr}(\alpha_r, 0) \qquad (1.8\text{ b})$$

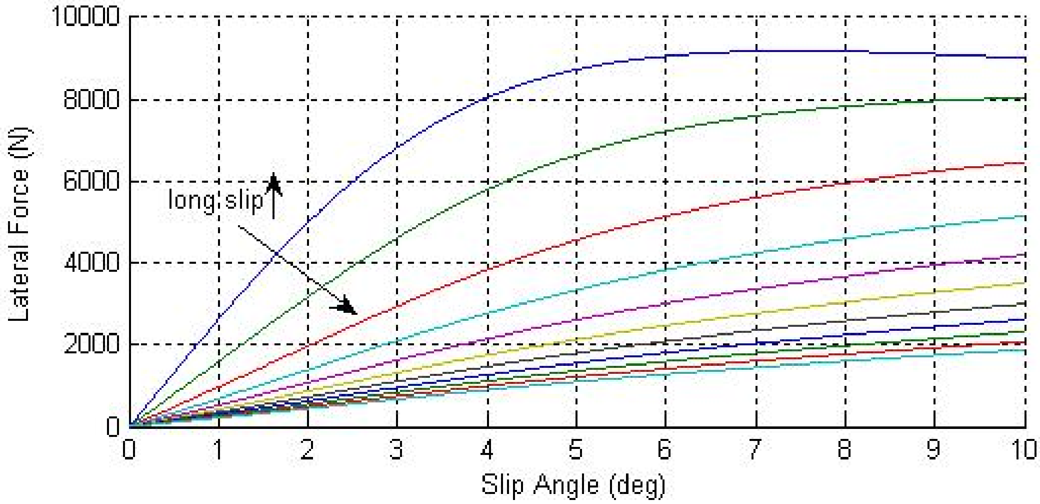
This assumption simplifies the identification problem, because it allowed us to estimate 4 scalar functions instead of 2 scalar fields. A 1-dimensional problem is much simpler than a 2-dimensional one. However this hypothesis is not completely true. In fact its consequences are the following ones:

1. In the identified plant, the rear longitudinal characteristic $F_{xr}(\alpha_r, k_r)$, even if α_r is variable, has its maximum always in the same point $k_{r,sat}$, while the peak value is simply $p_x(\alpha_r)F_{xr,\max}$.
2. in the identified plant, the rear lateral characteristic $F_{yr}(\alpha_r, k_r)$, even if k_r is variable, has its maximum always in the same point $\alpha_{r,sat}$, while the peak value is simply $p_y(k_r)F_{yr,\max}$.
3. Every point of these characteristics is "penalized" the same way and in consequence, the derivatives are "penalized" as well

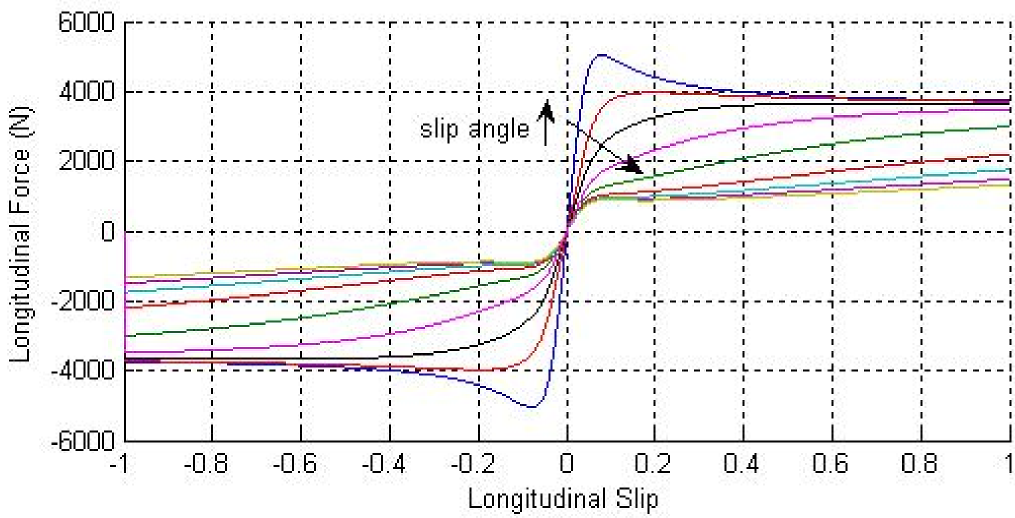
$$\frac{\partial}{\partial k_r} F_{xr}(\alpha_r, k_r) = p_x(\alpha_r) \frac{\partial}{\partial k_r} F_{xr}(0, k_r)$$

$$\frac{\partial}{\partial \alpha_r} F_{yr}(\alpha_r, k_r) = p_y(k_{rj}) \frac{\partial}{\partial \alpha_r} F_{yr}(\alpha_r, 0)$$

The behavior of the real plant is different:



Lateral Characteristic (vehicle)



Longitudinal Characteristic (vehicle)

and, in particular, the maximum of the characteristics increases if compared to the conditions of pure lateral or longitudinal slip. This difference in the behavior can cause of lack of matching between the plant model and the real one, in combined slip conditions, and also a worse behavior of the control action.

Chapter 3

Control System Design

3.1 Controller structure

Given the plant

$$\dot{x} = f(x) + G\ u$$

where:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} v_{yNS} \\ v_\psi \\ k_r \\ \delta \end{bmatrix}$$

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \delta_M \\ T_{AD} \end{bmatrix}$$

$$
\begin{aligned}
f(x) &= \begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \\ f_4(x) \end{bmatrix} = \\
&= \begin{bmatrix} f_1(x_1, x_2, x_3) \\ f_2(x_1, x_2, x_3, x_4) \\ f_3(x_1, x_2, x_3) \\ f_4(x_4) \end{bmatrix} = \\
&= \begin{bmatrix} -v_x x_2 + \frac{p_y(x_3) F_{yr}(-\frac{x_1-(l_r-l_{NS})x_2}{v_x}) l - p_x(-\frac{x_1-(l_r-l_{NS})x_2}{v_x}) F_{xr}(x_3) t}{m l_f} \\ \frac{F_{yf}(x_4 - \frac{x_1+(l_f+l_{NS})x_2}{v_x}) l_f - p_y(x_3) F_{yr}(-\frac{x_1-(l_r-l_{NS})x_2}{v_x}) l_r + p_x(-\frac{x_1-(l_r-l_{NS})x_2}{v_x}) F_{xr}(x_3) t}{J_z} \\ -K_2 \; p_x(-\frac{x_1-(l_r-l_{NS})x_2}{v_x}) F_{xr}(x_3) \\ -\frac{1}{\tau_M} x_4 \end{bmatrix}
\end{aligned}
$$

$$
G = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & K_3 \\ \frac{1}{\tau_M} & 0 \end{bmatrix}
$$

$$
K_2 = \frac{R_w^2}{J_w v_x}
$$

$$
K_3 = \frac{R_w}{J_w v_x}
$$

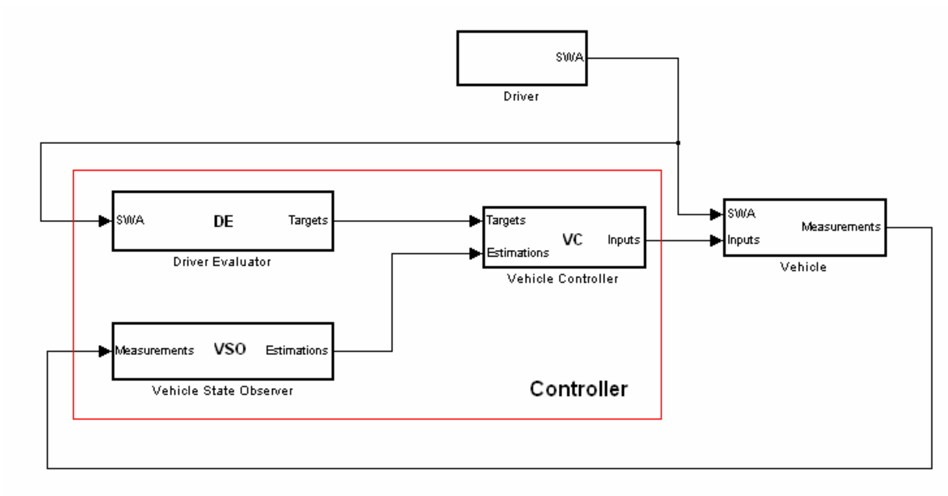
and 2 targets functions $v_{yNS}(t)$ and $v_\psi(t)$, for lateral velocity and yaw rate, the control problem (CP) is to find the control law $u(x)$ to insure:

1. global stability
2. lateral velocity and yaw rate tracking
3. comfort driving

The controller is composed by:

- DE, driver evaluator, to shape the target lateral velocity and yaw rate;
- VC, vehicle feedback controller, over which is the focus of the paper.

The measurements are yaw rate and steering angle. However, we can also consider lateral velocity as a known signal, by means of a VSO (vehicle state observer), over which this work does not focus.



Control system structure

3.2 Multiple Output Control - AFS and AD

We consider the following output function:

$$y = \begin{bmatrix} v_{yNS} \\ v_{\psi} \end{bmatrix} = C\ x$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

We differentiate the first output signal, until the input appears:

$$\dot{y}_1 = \dot{v}_{yNS} = f_1(x)$$

$$\begin{aligned}
\ddot{y}_1 &= \ddot{v}_{yNS} = \dot{f}_1(x) = \frac{df_1(x)}{dx}\dot{x} = \\
&= \frac{df_1(x)}{dx}\left(f(x) + Gu\right) = \\
&= \frac{df_1(x)}{dx}f(x) + \frac{df_1(x)}{dx}Gu = \\
&= \frac{df_1(x)}{dx}f(x) + \frac{1}{\tau_M}\frac{\partial f_1(x)}{\partial x_4}u_1 + K_3\frac{\partial f_1(x)}{\partial x_3}u_2 = \\
&= \frac{df_1(x)}{dx}f(x) + K_3\frac{\partial f_1(x)}{\partial x_3}u_2
\end{aligned}$$

since

$$\frac{\partial f_1(x)}{\partial x_4} = 0$$

Then we differentiate the second output signal, until the input appears:

$$\dot{y}_2 = \dot{v}_\psi = f_2(x)$$

$$\begin{aligned}
\ddot{y}_2 &= \ddot{v}_\psi = \dot{f}_2(x) = \frac{df_2(x)}{dx}\dot{x} = \\
&= \frac{df_2(x)}{dx}\left(f(x) + Gu\right) = \\
&= \frac{df_2(x)}{dx}f(x) + \frac{df_2(x)}{dx}Gu = \\
&= \frac{df_2(x)}{dx}f(x) + \frac{1}{\tau_M}\frac{\partial f_2(x)}{\partial x_4}u_1 + K_3\frac{\partial f_2(x)}{\partial x_3}u_2
\end{aligned}$$

The total relative degree of the non-linear system is $r = 4 = n$, where

n is the state-space dimension. So the situation is really convenient for the application of feedback linearization method, since there is no internal dynamics rendered "unobservable" in the linearization.

$$
\begin{aligned}
v &= \begin{bmatrix} \ddot{v}_{yNS} \\ \ddot{v}_{\psi} \end{bmatrix} = \\
&= \begin{bmatrix} \frac{df_1(x)}{dx} f(x) + K_3 \frac{\partial f_1(x)}{\partial x_3} u_2 \\ \frac{df_2(x)}{dx} f(x) + \frac{1}{\tau_M} \frac{\partial f_2(x)}{\partial x_4} u_1 + K_3 \frac{\partial f_2(x)}{\partial x_3} u_2 \end{bmatrix} = \\
&= F(x) + G(x)u
\end{aligned}
$$

where

$$
F(x) = \begin{bmatrix} \frac{df_1(x)}{dx} f(x) \\ \frac{df_2(x)}{dx} f(x) \end{bmatrix}
$$

$$
G(x) = \begin{bmatrix} 0 & K_3 \frac{\partial f_1(x)}{\partial x_3} \\ \frac{1}{\tau_M} \frac{\partial f_2(x)}{\partial x_4} & K_3 \frac{\partial f_2(x)}{\partial x_3} \end{bmatrix}
$$

So the control law

$$
u = G^{-1}(x) \left(v - F(x) \right)
$$

is applicable to the control problem and guarantees stability and reference tracking, if $G(x)$ is invertible.

The term v is imposed by a linear controller to get the desired performance (see the following paragraph).

In detail, the control inputs are:

$$
\begin{aligned}
u &= \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \\
&= G^{-1}(x) \begin{bmatrix} v_1 - \frac{df_1(x)}{dx} f(x) \\ v_2 - \frac{df_2(x)}{dx} f(x) \end{bmatrix} = \\
&= \begin{bmatrix} -\tau_M \frac{\frac{\partial f_2(x)}{\partial x_3}}{\frac{\partial f_1(x)}{\partial x_3} \frac{\partial f_2(x)}{\partial x_4}} & \tau_M \frac{1}{\frac{\partial f_2(x)}{\partial x_4}} \\ \frac{1}{K_3 \frac{\partial f_1(x)}{\partial x_3}} & 0 \end{bmatrix} \begin{bmatrix} v_1 - \frac{df_1(x)}{dx} f(x) \\ v_2 - \frac{df_2(x)}{dx} f(x) \end{bmatrix} = \\
&= \begin{bmatrix} -\tau_M \frac{\frac{\partial f_2(x)}{\partial x_3} \left(v_1 - \frac{df_1(x)}{dx} f(x) \right)}{\frac{\partial f_1(x)}{\partial x_3} \frac{\partial f_2(x)}{\partial x_4}} + \tau_M \frac{\left(v_2 - \frac{df_2(x)}{dx} f(x) \right)}{\frac{\partial f_2(x)}{\partial x_4}} \\ \frac{v_1 - \frac{df_1(x)}{dx} f(x)}{K_3 \frac{\partial f_1(x)}{\partial x_3}} \end{bmatrix}
\end{aligned}
$$

So the differential torque (AD) is

$$
T_{AD} = u_2 = \frac{v_1 - \frac{df_1(x)}{dx} f(x)}{\frac{R_w}{J_w v_x} \frac{\partial f_1(x)}{\partial x_3}}
$$

and it depends only on the input v_1 (related to the the lateral velocity tracking).

Substituting u_2 in the AFS control expression, we get:

$$
\delta_M = u_1 = \frac{v_2 - \frac{df_2(x)}{dx} f(x) - \frac{R_w}{J_w v_x} \frac{\partial f_2(x)}{\partial x_3} u_2}{\frac{1}{\tau_M} \frac{\partial f_2(x)}{\partial x_4}}
$$

The gradients assume the following expressions:

$$\frac{df_1(x)}{dx} = \begin{bmatrix} \frac{lp_y(x_3)}{ml_f}\frac{\partial F_{yr}}{\partial x_1} - \frac{tF_{xr}(x_3)}{ml_f}\frac{\partial p_x}{\partial x_1} \\ -v_x + \frac{lp_y(x_3)}{ml_f}\frac{\partial F_{yr}}{\partial x_2} - \frac{tF_{xr}(x_3)}{ml_f}\frac{\partial p_x}{\partial x_2} \\ \frac{lF_{yr}(-\frac{x_1-(l_r-l_{NS})x_2}{v_x})}{ml_f}\frac{\partial p_y}{\partial x_3} - \frac{tp_x(-\frac{x_1-(l_r-l_{NS})x_2}{v_x})}{ml_f}\frac{\partial F_{xr}}{\partial x_3} \\ 0 \end{bmatrix}^T$$

$$\frac{df_2(x)}{dx} = \begin{bmatrix} \frac{l_f}{J_z}\frac{\partial F_{yf}}{\partial x_1} - \frac{l_r p_y(x_3)}{J_z}\frac{\partial F_{yr}}{\partial x_1} + \frac{tF_{xr}(x_3)}{J_z}\frac{\partial p_x}{\partial x_1} \\ \frac{l_f}{J_z}\frac{\partial F_{yf}}{\partial x_2} - \frac{l_r p_y(x_3)}{J_z}\frac{\partial F_{yr}}{\partial x_2} + \frac{tF_{xr}(x_3)}{J_z}\frac{\partial p_x}{\partial x_2} \\ -\frac{l_r F_{yr}(-\frac{x_1-(l_r-l_{NS})x_2}{v_x})}{J_z}\frac{\partial p_y}{\partial x_3} + \frac{tp_x(-\frac{x_1-(l_r-l_{NS})x_2}{v_x})}{J_z}\frac{\partial F_{xr}}{\partial x_3} \\ \frac{l_f}{J_z}\frac{\partial F_{yf}}{\partial x_4} \end{bmatrix}^T$$

3.2.1 The linearized system

The technique of feedback linearization transforms the vehicle non-linear plant $\dot{x} = f(x) + G\ u$ into an equivalent linear time-invariant dynamics, in the familiar form

$$\dot{z} = A\ z + b\ v$$

where

$$z = \begin{bmatrix} v_{yNS} \\ \dot{v}_{yNS} \\ v_\psi \\ \dot{v}_\psi \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$b = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

The imposed linear control law is (see Appendix B):

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \ddot{v}_{yNS,T} + K_{D1}\left(\dot{v}_{yNS,T} - \dot{v}_{yNS}\right) + K_{P1}\left(v_{yNS,T} - v_{yNS}\right) \\ \ddot{v}_{\psi,T} + K_{D2}\left(\dot{v}_{\psi,T} - \dot{v}_{\psi}\right) + K_{P2}\left(v_{\psi,T} - v_{\psi}\right) \end{bmatrix}$$

where the constants K_{D1}, K_{D2}, K_{P1}, K_{P2} are tuned such that the polynomials

$$p^2 + K_{D1}\, p + K_{P1}$$

$$p^2 + K_{D2}\, p + K_{P2}$$

have all their roots strictly in the left-hand plane.

3.2.2 Invertibility of the transformation

$G(x)$ is not invertible if one of the following conditions is verified:

$$\frac{\partial f_1(x)}{\partial x_3} = 0 \text{ or } \frac{\partial f_2(x)}{\partial x_4} = 0$$

$$\frac{\partial f_1(x)}{\partial x_3} = \frac{F_{yr}(-\frac{x_1-(l_r-l_{NS})x_2}{v_x})l}{ml_f}\frac{dp_y(x_3)}{dx_3} - \frac{p_x(-\frac{x_1-(l_r-l_{NS})x_2}{v_x})t}{ml_f}\frac{dF_{xr}(x_3)}{dx_3}$$

$$\frac{\partial f_2(x)}{\partial x_4} = \frac{l_f}{J_z}\frac{\partial F_{yf}(x_4 - \frac{x_1+(l_f+l_{NS})x_2}{v_x})}{\partial x_4}$$

$$\begin{aligned}\frac{\partial f_1(x)}{\partial x_3} &= 0 \Leftrightarrow \\ &\Leftrightarrow F_{yr}(-\frac{x_1-(l_r-l_{NS})x_2}{v_x})l\frac{dp_y(x_3)}{dx_3} = \\ &= p_x(-\frac{x_1-(l_r-l_{NS})x_2}{v_x})t\frac{dF_{xr}(x_3)}{dx_3}\end{aligned}$$

$$\begin{aligned}\frac{\partial f_2(x)}{\partial x_4} &= 0 \Leftrightarrow \\ &\Leftrightarrow \frac{\partial F_{yf}(x_4-\frac{x_1+(l_f+l_{NS})x_2}{v_x})}{\partial x_4} = 0\end{aligned}$$

Since

$$\alpha_f = x_4 - \frac{x_1+(l_f+l_{NS})x_2}{v_x}$$

then

$$\frac{\partial F_{yf}(x_4-\frac{x_1+(l_f+l_{NS})x_2}{v_x})}{\partial x_4} = \frac{dF_{yf}(\alpha_f)}{d\alpha_f}\frac{\partial \alpha_f}{\partial x_4} = \frac{dF_{yf}(\alpha_f)}{d\alpha_f}$$

So the first condition for the invertibility is:

$$\frac{dF_{yf}(\alpha_f)}{d\alpha_f} \neq 0$$

Since

$$\alpha_r = -\frac{x_1-(l_r-l_{NS})x_2}{v_x}$$

the second condition is:

$$\frac{\partial f_1(x)}{\partial x_3} \neq 0 \Leftrightarrow lF_{yr}(\alpha_r)\frac{dp_y(k_r)}{dk_r} \neq tp_x(\alpha_r)\frac{dF_{xr}(k_r)}{dk_r}$$

3.3 Single Output Control - only AFS

The performance of the Multiple Output Control can be compared to the performance of a Single Output Control, using only the Active Front Steering actuator in a yaw rate tracking problem.

We consider the vehicle non-linear plant

$$\dot{x} = f(x) + G\ u$$

where x, f, G, u have been defined in the previous part.

We decide to use only the steering control, so:

$$T_{AD} = u_2 = 0$$

The consequence is that, if $k_r(t_0) = 0$

$$k_r(t) = 0 \qquad \forall t \geq t_0$$

$$F_{xr}(k_r(t)) = F_{xr}(0) = 0 \qquad \forall t \geq t_0$$

$$p_y(k_r(t)) = p_y(0) = 1 \qquad \forall t \geq t_0$$

So we can consider a simplified plant, with only 3 state variables, 1 input

$$\dot{x} = f(x) + G\ u$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} v_{yNS} \\ v_\psi \\ \delta \end{bmatrix}$$

$$u = \delta_M$$

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{bmatrix} = \begin{bmatrix} f_1(x_1,x_2) \\ f_2(x_1,x_2,x_3) \\ f_3(x_3) \end{bmatrix} =$$

$$= \begin{bmatrix} -v_x x_2 + \frac{F_{yr}(-\frac{x_1-(l_r-l_{NS})x_2}{v_x})l}{ml_f} \\ \frac{F_{yf}(x_3-\frac{x_1+(l_f+l_{NS})x_2}{v_x})l_f - F_{yr}(-\frac{x_1-(l_r-l_{NS})x_2}{v_x})l_r}{J_z} \\ -\frac{1}{\tau_M}x_3 \end{bmatrix}$$

$$G = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{\tau_M} & 0 \end{bmatrix}$$

and a target function $v_\psi(t)$, for yaw rate.
We consider the following output function:

$$y = v_\psi$$

We differentiate the output signal, until the input appears:

$$\dot{y} = \dot{v}_\psi = f_2(x)$$

$$\begin{aligned} \ddot{y}_2 &= \ddot{v}_\psi = \dot{f}_2(x) = \frac{df_2(x)}{dx}\dot{x} = \\ &= \frac{df_2(x)}{dx}(f(x) + Gu) = \\ &= \frac{df_2(x)}{dx}f(x) + \frac{df_2(x)}{dx}Gu = \\ &= \frac{df_2(x)}{dx}f(x) + \frac{1}{\tau_M}\frac{\partial f_2(x)}{\partial x_3}u \end{aligned}$$

The relative degree of the non-linear system is $r = 2 < n = 3$, where

n is the state-space dimension. So the situation is really convenient for the application of feedback linearization method, but there is an internal dynamics rendered "unobservable" in the linearization.

$$v = \ddot{v}_\psi = \frac{df_2(x)}{dx} f(x) + \frac{1}{\tau_M} \frac{\partial f_2(x)}{\partial x_3} u$$

So the control law

$$\delta_M = u = \frac{v - \frac{df_2(x)}{dx} f(x)}{\frac{1}{\tau_M} \frac{\partial f_2(x)}{\partial x_3}}$$

is applicable to the control problem and guarantees stability and reference tracking, if the transformation is invertible and the internal dynamics is stable.

The term v is imposed, as in the multiple output control, by a linear controller to get the desired performance.

If we compare this expression with that one in the previous part, we can notice that it is the same control law, with $\frac{\partial f_2(x)}{\partial x_3} = 0$ or $u_2 = T_{AD} = 0$.

3.3.1 The linearized system

The technique of feedback linearization transforms the vehicle non-linear plant $\dot{x} = f(x) + G\ u$ into an equivalent linear time-invariant dynamics, in the familiar form

$$\dot{\xi} = A\ \xi + b\ v$$

where

$$\xi = \begin{bmatrix} v_\psi \\ \dot{v}_\psi \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The linear control law is imposed such that the polynomial

$$p^2 + K_D\ p + K_P$$

has all its roots strictly in the left-hand plane.

3.3.2 Invertibility of the transformation

We should study the stability of the internal dynamics, in order to guarantee the stability of the controlled system. However, we will consider an intrinsic property of the dynamic system, called zero-dynamics (see Appendix B).

We consider a transformation

$$x \Longleftrightarrow z$$

where z is composed of a linearized (controlled) part ξ, and of a unobservable part η.

$$z = \begin{bmatrix} \eta \\ \xi \end{bmatrix}$$

$$\xi = \begin{bmatrix} v_\psi \\ \dot{v}_\psi \end{bmatrix}$$

We choose η conveniently and verify if the transformation is invertible.

$$\eta = v_{yNS}$$

$$z = \begin{bmatrix} v_{yNS} \\ v_\psi \\ \dot{v}_\psi \end{bmatrix} = \Phi(x) = \begin{bmatrix} x_1 \\ x_2 \\ f_2(x_1, x_2, x_3) \end{bmatrix}$$

The transformation is invertible if and only if:

$$\begin{aligned} \left|\frac{d\Phi}{dx}\right| &= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\partial f_2(x)}{\partial x_1} & \frac{\partial f_2(x)}{\partial x_2} & \frac{\partial f_2(x)}{\partial x_3} \end{vmatrix} \neq 0 \Longleftrightarrow \\ &\Longleftrightarrow \left|\frac{\partial f_2(x)}{\partial x_3}\right| \neq 0 \Longleftrightarrow \frac{dF_{yf}(\alpha_f)}{d\alpha_f} \neq 0 \end{aligned}$$

So we find only one of the 2 conditions for invertibility of the MIMO system:

$$\frac{dF_{yf}(\alpha_f)}{d\alpha_f} \neq 0$$

3.3.3 Stability of zero-dynamics

The stability of the zero-dynamics $\eta = v_{yNS}$ is a necessary (not sufficient) condition for the stability of the controlled system in reference tracking conditions.

So we consider the stability of v_{yNS} with

$$\begin{aligned} v &= 0 \\ \xi &= 0 \end{aligned}$$

i.e. the stability of the internal dynamics when the observable part ξ has already been stabilized.

The dynamic equation is:

$$\dot{x}_1 = -v_x x_2 + \frac{F_{yr}(-\frac{x_1-(l_r-l_{NS})x_2}{v_x})l}{ml_f} = \frac{F_{yr}(-\frac{x_1}{v_x})l}{ml_f}$$

since $x_2 = 0$.
The tyre function F_{yr} has the following property:

$$\begin{aligned} F_{yr}(\alpha_r) &> 0 \iff \alpha_r > 0 \\ F_{yr}(\alpha_r) &= 0 \iff \alpha_r = 0 \\ F_{yr}(\alpha_r) &< 0 \iff \alpha_r < 0 \end{aligned}$$

so

$$\begin{aligned} x_1 &> 0 \implies \dot{x}_1 < 0 \\ x_1 &= 0 \implies \dot{x}_1 = 0 \\ x_1 &< 0 \implies \dot{x}_1 > 0 \end{aligned}$$

and it is proved that v_{yNS} is asymptotically stable.

3.4 Single Output Control - AD only

A trivial SISO controller, using only the Active Differential, can be derived from the previous MIMO one, imposing

$$\delta_C = 0$$

that is equivalent to turning the AFS off.
Since the lateral velocity tracking control is independent from u_1 in the

MIMO controller expression

$$T_{AD} = u_2 = \frac{v_1 - \frac{df_1(x)}{dx} f(x)}{K_3 \frac{\partial f_1(x)}{\partial x_3}}$$

this control provides tracking of the lateral velocity, without taking care of the yaw rate tracking.

However, if only one actuator is available, it is usually used to solve a problem of tracking a desired yaw rate trajectory.

So we consider the simplified system

$$\dot{x} = f(x) + G\ u$$

where:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} v_{yNS} \\ v_\psi \\ k_r \end{bmatrix}$$

$$u = T_{AD}$$

$$f(x) = \begin{bmatrix} -v_x x_2 + \frac{p_y(x_3) F_{yr}(-\frac{x_1-(l_r-l_{NS})x_2}{v_x}) l - p_x(-\frac{x_1-(l_r-l_{NS})x_2}{v_x}) F_{xr}(x_3) t}{m l_f} \\ \frac{F_{yf}(\delta_D - \frac{x_1+(l_f+l_{NS})x_2}{v_x}) l_f - p_y(x_3) F_{yr}(-\frac{x_1-(l_r-l_{NS})x_2}{v_x}) l_r + p_x(-\frac{x_1-(l_r-l_{NS})x_2}{v_x}) F_{xr}(x_3) t}{J_z} \\ -\frac{R_w^2}{J_w v_x} p_x(-\frac{x_1-(l_r-l_{NS})x_2}{v_x}) F_{xr}(x_3) \end{bmatrix}$$

$$G = \begin{bmatrix} 0 \\ 0 \\ \frac{R_w}{J_w v_x} \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \dot{v}_{yNS} \\ \dot{v}_\psi \\ \dot{k}_r \end{bmatrix} = \begin{bmatrix} -v_x x_2 + \frac{p_y(x_3)F_{yr}(-\frac{x_1-(l_r-l_{NS})x_2}{v_x})l - p_x(-\frac{x_1-(l_r-l_{NS})x_2}{v_x})F_{xr}(x_3)t}{ml_f} \\ \frac{F_{yf}(\delta_D - \frac{x_1+(l_f+l_{NS})x_2}{v_x})l_f - p_y(x_3)F_{yr}(-\frac{x_1-(l_r-l_{NS})x_2}{v_x})l_r + p_x(-\frac{x_1-(l_r-l_{NS})x_2}{v_x})F_{xr}(x_3)t}{J_z} \\ -\frac{R_w^2}{J_w v_x} p_x(-\frac{x_1-(l_r-l_{NS})x_2}{v_x})F_{xr}(x_3) + \frac{R_w}{J_w v_x}u \end{bmatrix}$$

and the following output function

$$y = x_2 = v_\psi$$

We differentiate the output signal, until the input appears:

$$\dot{y} = \dot{v}_\psi = f_2(x)$$

$$\begin{aligned} \ddot{y} &= \ddot{v}_\psi = \dot{f}_2(x) = \frac{df_2(x)}{dx}\dot{x} = \\ &= \frac{df_2(x)}{dx}\left(f(x) + Gu\right) = \\ &= \frac{df_2(x)}{dx}f(x) + \frac{df_2(x)}{dx}Gu = \\ &= \frac{df_2(x)}{dx}f(x) + \frac{R_w}{J_w v_x}\frac{\partial f_2(x)}{\partial x_3}u \end{aligned}$$

The relative degree of the non-linear system is $r = 2 < n = 3$, where n is the state-space dimension. So the situation is really convenient for the application of feedback linearization method, but there is an internal dynamics rendered "unobservable" in the linearization.

$$v = \ddot{v}_\psi = \frac{df_2(x)}{dx}f(x) + \frac{R_w}{J_w v_x}\frac{\partial f_2(x)}{\partial x_3}u$$

So the control law

$$T_{AD} = u = \frac{v - \frac{df_2(x)}{dx}f(x)}{\frac{R_w}{J_w v_x}\frac{\partial f_2(x)}{\partial x_3}}$$

is applicable to the control problem and guarantees stability and reference tracking, if the transformation is invertible and the internal dynamics is stable.

The term v is imposed, as in the AFS-only output control, by a linear controller to get the desired performance. The nonlinear plant is transformed into an equivalent linear time-invariant dynamics, but we have to study the stability of the zero-dynamics.

Considering the same state transformation of the AFS case:

$$x \Longleftrightarrow z$$

$$z = \begin{bmatrix} \eta \\ \xi \end{bmatrix}$$

$$\xi = \begin{bmatrix} v_\psi \\ \dot{v}_\psi \end{bmatrix}$$

$$\eta = v_{yNS}$$

$$z = \begin{bmatrix} v_{yNS} \\ v_\psi \\ \dot{v}_\psi \end{bmatrix} = \Phi(x) = \begin{bmatrix} x_1 \\ x_2 \\ f_2(x_1, x_2, x_3) \end{bmatrix}$$

The transformation is invertible if and only if:

$$\begin{aligned} \left|\frac{d\Phi}{dx}\right| &= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\partial f_2(x)}{\partial x_1} & \frac{\partial f_2(x)}{\partial x_2} & \frac{\partial f_2(x)}{\partial x_3} \end{vmatrix} \neq 0 \Longleftrightarrow \\ &\Longleftrightarrow \left|\frac{\partial f_2(x)}{\partial x_3}\right| \neq 0 \Longleftrightarrow \\ &\Longleftrightarrow l_r F_{yr}(\alpha_r) \frac{\partial p_y(k_r)}{\partial k_r} \neq t p_x(\alpha_r) \frac{\partial F_{xr}(k_r)}{\partial k_r} \end{aligned}$$

This condition is very similar to the invertibility condition $\frac{\partial f_1(x)}{\partial x_3} \neq 0$ of the MIMO control system

$$lF_{yr}(\alpha_r)\frac{dp_y(k_r)}{dk_r} \neq tp_x(\alpha_r)\frac{dF_{xr}(k_r)}{dk_r}$$

In particular, the following implication is always verified

$$\frac{\partial f_1(x)}{\partial k_r} < 0 \Longrightarrow \frac{\partial f_2(x)}{\partial k_r} > 0$$

where $f_1(x)$ and $f_2(x)$ are the state functions for v_{yNS} and v_ψ.

The zero dynamics v_{yNS}, imposing

$$\begin{aligned} v &= 0 \\ \xi &= 0 \end{aligned}$$

is asymptotically stable. The proof is the same as in the AFS-only case.

3.5 Intuitive meaning of the control law

The MIMO control law, given by the feedback linearization method, has the following expression

$$T_{AD} = \frac{v_1 - \frac{df_1(x)}{dx} f(x)}{\frac{R_w}{J_w v_x} \frac{\partial f_1(x)}{\partial x_3}} \qquad \text{(MIMO AD)}$$

$$\delta_M = \frac{v_2 - \frac{df_2(x)}{dx} f(x) - \frac{R_w}{J_w v_x} \frac{\partial f_2(x)}{\partial x_3} T_{AD}}{\frac{1}{\tau_M} \frac{\partial f_2(x)}{\partial x_4}} \qquad \text{(MIMO AFS)}$$

The SISO (AFS only) control law is

$$\delta_M = \frac{v - \frac{df_2(x)}{dx} f(x)}{\frac{1}{\tau_M} \frac{\partial f_2(x)}{\partial x_3}} \qquad \text{(AFS-only)}$$

The SISO (AD only) control law is

$$T_{AD} = \frac{v - \frac{df_1(x)}{dx} f(x)}{\frac{R_w}{J_w v_x} \frac{\partial f_1(x)}{\partial x_3}} \qquad \text{(MIMO AD)}$$

in lateral velocity tracking, while it is

$$T_{AD} = u = \frac{v - \frac{df_2(x)}{dx} f(x)}{\frac{R_w}{J_w v_x} \frac{\partial f_2(x)}{\partial x_3}} \qquad \text{(AD-only)}$$

to solve a yaw rate tracking control problem.

These expressions have a definite structure and a definite meaning. Some considerations about that are the following ones

1. First of all, the AD only controller for lateral velocity tracking is a trivial extension of the MIMO case, without AFS, and so we no longer consider it.

2. The numerators of these expressions are made the following way

$$v - \frac{df_j(x)}{dx} f(x)$$

where $f_j(x)$ is the *state function* of the state over which the tracking control is focused, and v is the term given by the linear controller

$$\ddot{x}_{j,T} + K_{Dj} \left(\dot{x}_{j,T} - \dot{x}_j \right) + K_{Pj} \left(x_{j,T} - x_j \right)$$

If we consider the meaning of the *state function*

$$\frac{df_j(x)}{dx} f(x) = \dot{f}_j(x) \mid_{u=0} = \ddot{x}_j \mid_{u=0}$$

the numerator of each control law can be written

$$\begin{aligned} \ddot{x}_j(u &= 0) - \ddot{x}_{j,T} + K_{Dj}\,(\dot{x}_{j,T} - \dot{x}_j) + K_{Pj}\,(x_{j,T} - x_j) = \\ &= \ddot{e}_j + K_{Dj}\ \dot{e}_j + K_{Pj}\ e_j \end{aligned}$$

3. The numerator of δ_M in the MIMO controller is the real example of integrated controller. In fact it adds to the structure $v - \frac{df_j(x)}{dx} f(x)$, discussed above, a term

$$\frac{R_w}{J_w v_x} \frac{\partial f_2(x)}{\partial k_r} T_{AD}$$

which takes care of the indirect influence of the AD control action on the yaw rate evolution. The meaning of the derivative and of the constant is discussed in 5. The influence of the AD control action on the yaw rate evolution is caused by the variation of the longitudinal slip due to the Active Differential. So, in the AFS-only, this term is obviously zero.

4. The denominators of the control law expressions are made this way

$$C_i \frac{\partial f_j(x)}{\partial x_i}$$

where $f_j(x)$ is the *state function* of the state over which the tracking control is focused (as in the numerator), x_i is referred to the state directly influenced by the actuator

$$x_i = \begin{cases} \delta & \text{(AFS)} \\ \\ k_r & \text{(AD)} \end{cases}$$

and C_i are constant

$$C_i = \begin{cases} \frac{1}{\tau_M} & \text{(AFS)} \\ \\ \frac{R_w}{J_w v_x} & \text{(AD)} \end{cases}$$

which are the linear terms present in the dynamic equations

$$\dot{x}_i = f_i(x) + C_i x_i$$

5. The derivatives $\frac{\partial f_j(x)}{\partial x_i}$ are responsible for the lack of applicability of the control action, in some conditions (see previous sections). In particular $\frac{\partial f_j(x)}{\partial x_i}$, for the AFS, coincides with the stiffness of the front lateral characteristic; so, the higher this stiffness is, the higher the effect of the AFS correction on the yaw rate dynamics is . For the AD, it is the factor which influences the effect of the AD correction on the dynamics of the neutral steering point lateral velocity.

In conclusion, the feedback linearization control law, in order to correct the tracking of the state x_j, provides an action with some features:

- The control action is directly proportional to the tracking error and to its first and second time derivative, according to the constant values K_P and K_D of the linear controller. The error in the second derivative is calculated as a difference between the desired value (output of the reference plant) and the value occurring without control action. The non-linear controller fulfils its function in this term.

- The control action is inversely proportional to some derivatives, multiplied by some coefficients. These quantities express the instantaneous influence of each actuator on the dynamics of the state, over which the tracking control is focused. It means that, the less this influence (the derivative) is, the higher the control action has to be to achieve the desired effect.

- Finally, in cases of integrated control, the control law, for each actuator, takes care of the indirect influence of the control action of the other actuators on the evolution of the states.

Chapter 4

Control System Implementation

4.1 Matlab-Simulink Environment

In the implementation of the control system, the Matlab/Simulink Platform was used. SIMULINK uses a graphical user interface (GUI) for solving process simulations. Instead of writing MATLAB code, we simply connect the necessary "icons" together to construct the block diagram. The "icons" represent possible inputs to the system, parts of the systems, or outputs of the system. SIMULINK allows the user to easily simulate systems of linear and nonlinear ordinary differential equations.

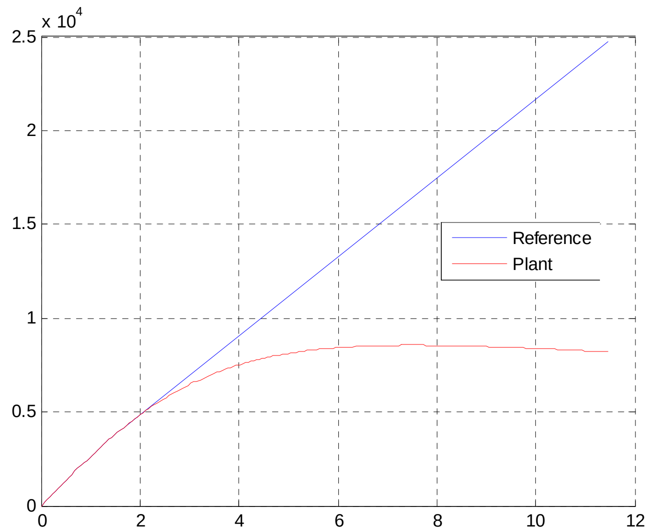
MATLAB has several different functions (built-ins) for the numerical solution of ordinary differential equations (ODE). The basic steps are typically the same. In the simulations, the function so-called "ode45" is used.

Ode45 is based on an explicit Runge-Kutta formula. That means the numerical solver ode45 combines a fourth order method and a fifth order method, both of which are similar to the classical fourth order Runge-Kutta (RK) method. The modified RK varies the step size, choosing the step size at each step in an attempt to achieve the desired accuracy. Therefore, the solver ode45 is suitable for a wide variety of initial value problems in practical applications. In general, ode45 is the best function to apply for most problems.

4.2 Reference

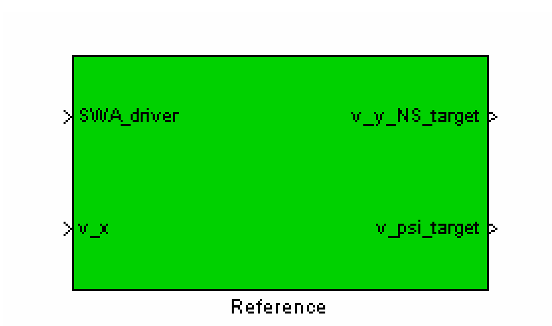
The reference signals for yaw rate and lateral velocity represent what the driver expects from the vehicle, given a manoeuvre (SWA). So, the reference is a plant, whose tyre characteristics are quite stiff, so that the vehicle can respond in an agile and quick way to every reasonable situation. The reference cannot be unstable because the driver does not expect an unstable behavior from the vehicle.

So, since the main feature of an ideal vehicle is the stability, the rear axle characteristic of the reference model will not have the unstable part of the characteristic; moreover, the stable part has a higher slope than the plant one.



Reference Rear Lateral Characteristic

The reference plant, or driver evaluator, through the front characteristic and the modified rear characteristic, calculates reference signals for yaw rate and lateral velocity, which get, in turn, as inputs into the linear controller.



Driver Evaluator

4.3 Applicability conditions

The control law given by the Feedback Linearization Method is not applicable when the decoupling matrix is not invertible, i.e.:

$$\frac{dF_{yf}(\alpha_f)}{d\alpha_f} = 0$$

or

$$lF_{yr}(\alpha_r)\frac{dp_y(k_r)}{dk_r} - tp_x(\alpha_r)\frac{dF_{xr}(k_r)}{dk_r} = 0$$

The first condition is a condition on the front axle, the second one is on the rear axle.

Actually, in the implementation, there is a neighborhood of these conditions on the states, in which the control law assumes really high values, and so it cannot be provided by real actuators. In these conditions, we need to modify the control law.

In this way, the perfect output tracking is not performed anymore, but the stability of the vehicle is guaranteed.

Moreover, we do not want to guarantee tracking in the instability region of the characteristics, but we need the control act to prevent instability. Because of this consideration, our interest is in keeping control in the region

$$\begin{aligned} \frac{dF_{yf}(\alpha_f)}{d\alpha_f} &> 0 \\ \frac{dF_{yr}(\alpha_r)}{d\alpha_r} &> 0 \\ \frac{dF_{xr}(k_r)}{dk_r} &> 0 \end{aligned}$$

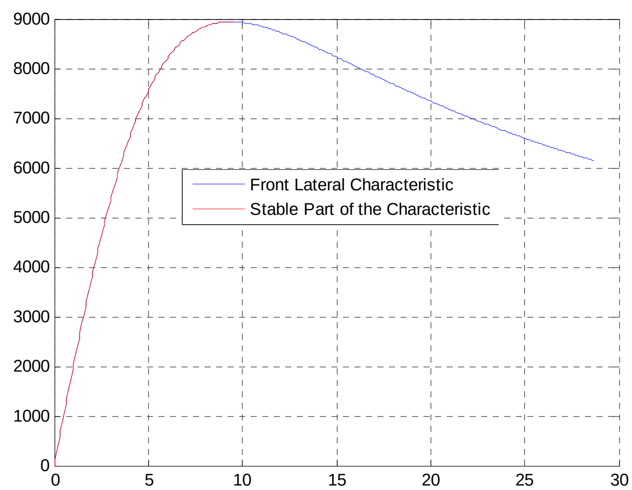
4.3.1 Front axle

The congruence equations provide an algebraic constraint between the front slip angle and the control

$$\alpha_f = \delta - \frac{v_y + l_f v_\psi}{v_x} = \delta_C + \delta_D - \frac{v_y + l_f v_\psi}{v_x} \quad (1.4\text{ a})$$

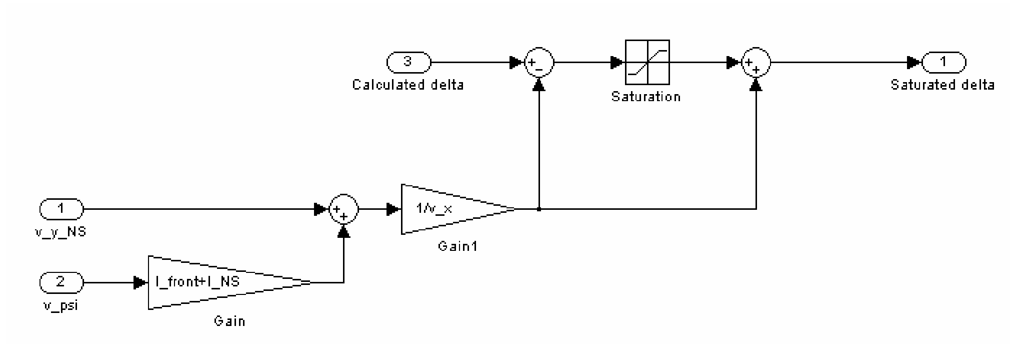
$$\delta_C = \alpha_f - \delta_D + \frac{v_y + l_f v_\psi}{v_x}$$

so we saturate the control to prevent that it drives the vehicle towards the instability zone of $F_{yf}(\alpha_f)$



Stable Part of Fy Front

This limitation is implemented according to the following scheme



Saturation scheme

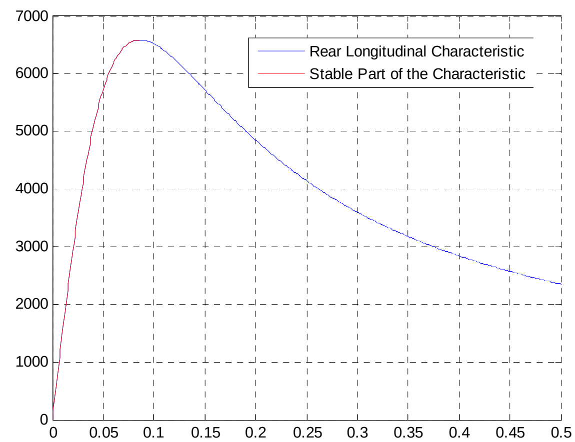
4.3.2 Rear Axle

In order to prevent saturations on the rear axle, a sort of penalty on the calculated torque has been implemented. In particular, it is a combination of two separable components:

1. When the rear right longitudinal slip reaches values near to the saturation point of the rear longitudinal characteristic, the calculated torque is multiplied by a factor $p_1 \in [0, 1]$. In the closeness of the saturation point, i.e. $k_r \approx 0.075$, p_1 is approximately set to zero, so that the slip equation simply becomes

$$\dot{k}_r = -\frac{R_w^2}{J_w v_x} F_{xrr}(\alpha_r, k_r)$$

   k_r and $\dot{k}_r$ have opposite signs, so that $|k_r|$ decreases, and the system moves away from the saturation condition.



Stable Part of Fx Rear

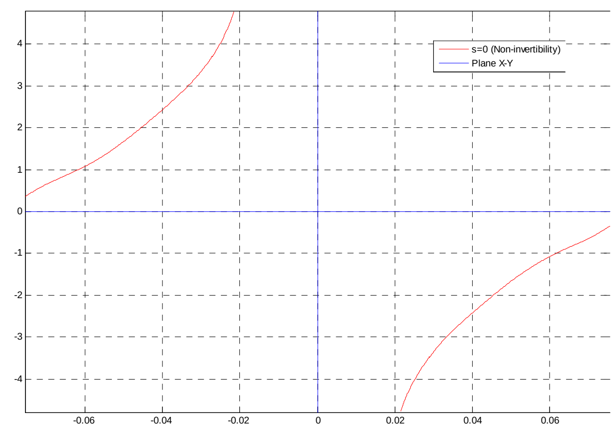
2. When the combination (α_r, k_r) is such that the expression

$$lF_{yr}(\alpha_r)\frac{dp_y(k_r)}{dk_r} - tp_x(\alpha_r)\frac{dF_{xr}(k_r)}{dk_r}$$

assumes small values, similarly to the previous case, another factor $p_2 \in [0, 1]$ reduces the torque, preventing the system from reaching the singularity condition

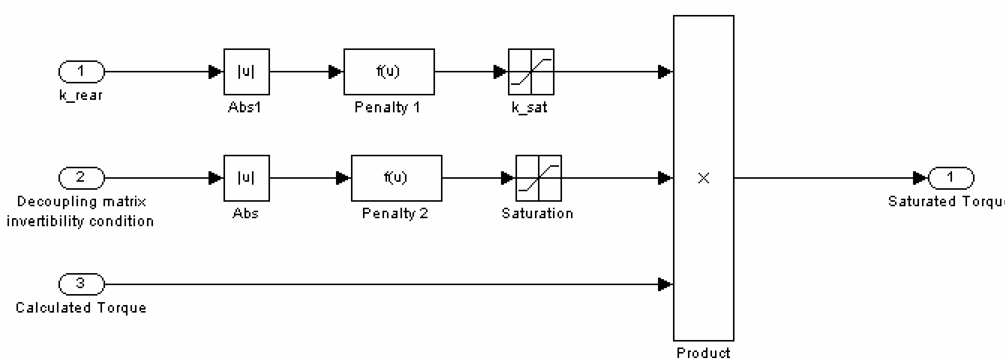
$$s(\alpha_r, k_r) = lF_{yr}(\alpha_r)\frac{dp_y(k_r)}{dk_r} - tp_x(\alpha_r)\frac{dF_{xr}(k_r)}{dk_r} = 0$$

Such condition is represented in the following plot. The central zone is the good one for the control action.



Good zone for the control action

3. Finally, the implemented scheme is the following one, in which both of the penalty functions and the calculated torque multiply each other, and the product is the applied torque



Saturation scheme

$$T_{app} = p_1(k_r)\ p_2(\alpha_r, k_r)\ T_{calc}$$

4.4 Alternative solutions

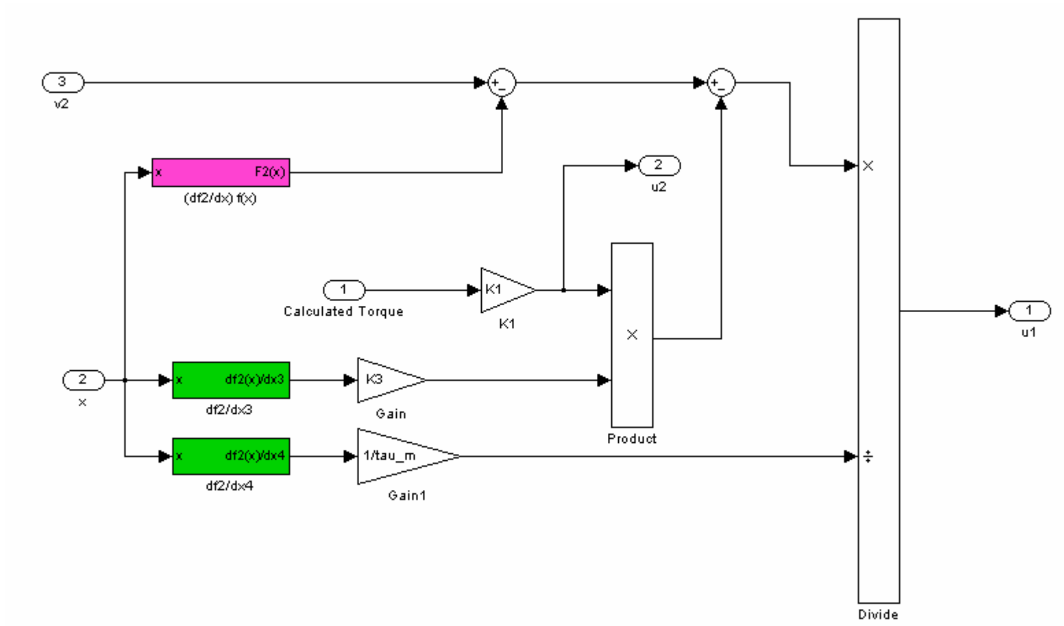
In the previous sections, possible solutions to the saturation problem have been described. However, other solutions were implemented, but they were not included in the final version of the controller.

4.4.1 Fixed gain on the torque control action

Some simulations (see Chapter 5) show that the active differential action causes a reduced action of the AFS, with the result that the front axle saturates, in the MIMO system, more often than in the SISO system.

A possibility to prevent this problem is to multiply the calculated torque for a gain that is decided "a priori"

$$T_{app} = K_1\ T_{calc}$$



Fixed Gain

In order to balance the reduced action of the AD, the AFS action has to be consequently modified

$$\delta_M = u_1 = \frac{v_2 - \frac{df_2(x)}{dx} f(x) - K_3 \frac{\partial f_2(x)}{\partial x_3} K_1 u_2}{\frac{1}{\tau_M} \frac{\partial f_2(x)}{\partial x_4}}$$

Varying the factor K_1 in $[0, 1]$ we get a set of different MIMO controllers. When K_1 is decreased, the action of the AD is less intrusive. With $K_1 = 0$, the control law is

$$T_{app} = u_2 = 0$$

$$\delta_M = u_1 = \frac{v_2 - \frac{df_2(x)}{dx} f(x)}{\frac{1}{\tau_M} \frac{\partial f_2(x)}{\partial x_4}}$$

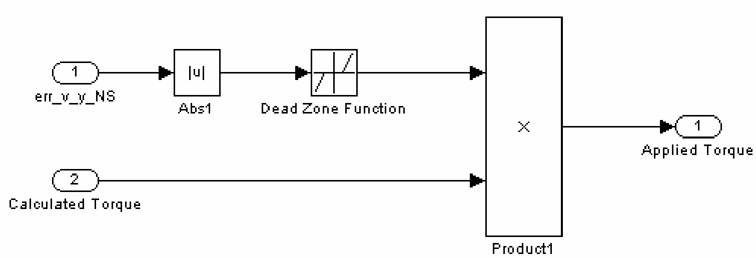
that is exactly the SISO (AFS only) control law.

4.4.2 Dead Zone

In some conditions, for example ramp steer manoeuvres, or constant radius turns, the stability can be guaranteed even by a passive system or by a SISO system, because it is not a really hard manoeuvre. The integrated controller, instead, by its simultaneous correction of lateral velocity and yaw rate, proves to be too intrusive for the driver perception.

So, in some manoeuvres, the action of the differential can be limited, because it is not really interesting a perfect tracking of the lateral velocity. This action can be achieved by "dead zones". It means that, if the difference between the measured value of v_{yNS} and the reference value $v_{yNS,T}$ is less than a tolerance band (dead band), the control action is not provided.

When the control acts in the dead band, e_{vyNS} is set to zero, so that the differential does not correct the error ($u_2 = 0$) and the MIMO control system behaves as the SISO (AFS only) one. In harder manoeuvres, on the contrary, as double step steer, the v_{yNS} error goes beyond the tolerance value, and so the active differential gradually starts giving its contribution to the tracking control, reducing the difference between the reference value and the measured one.



Dead Zone

Chapter 5

Simulations

This chapter shows the simulations of the present work. We distinguish two kinds of simulation environment:

- simulations in nominal conditions
- simulation in not nominal conditions

At first, the designed control system is tried on the nominal identified plant. It means that, except for possible saturations of actuators, we expect a very good behavior of the system. In fact the Feedback Linearization Method should perfectly cancel the nonlinearities of the nonlinear system, so that the closed-loop dynamics is in a linear form. On the linearized plant, a desired linear dynamics is imposed.

These conditions are nominal because there is no unmodeled dynamics in the system, and the parameters are perfectly known.

When one of this conditions is perturbed, we pass to a non-nominal environment of simulation. In the present case, we analyze two non-nominal conditions

- control in presence of perturbations of the identified plant
- control of the real vehicle

In these conditions, no robustness is guaranteed for the control system. An example of perturbation on the ideal plant is the presence of a strong wind. This is an unmodeled dynamics, which is not taken into account by the control system, and so it can yield undesired behavior.

However, the most critical situation is the simulation of control on the real vehicle model, with 6 degrees of freedom. This environment is really far from the ideal one, mainly because of the following conditions:

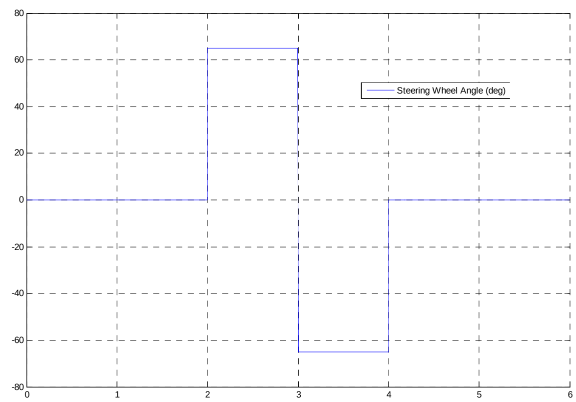
- unmodedel dynamics
- parameter uncertainty
- errors in the measurement of the vehicle variables

In such simulation environment, neither optimal tracking nor stability is guaranteed by the input-output linearization theoretical results.

5.1 Nominal Environment

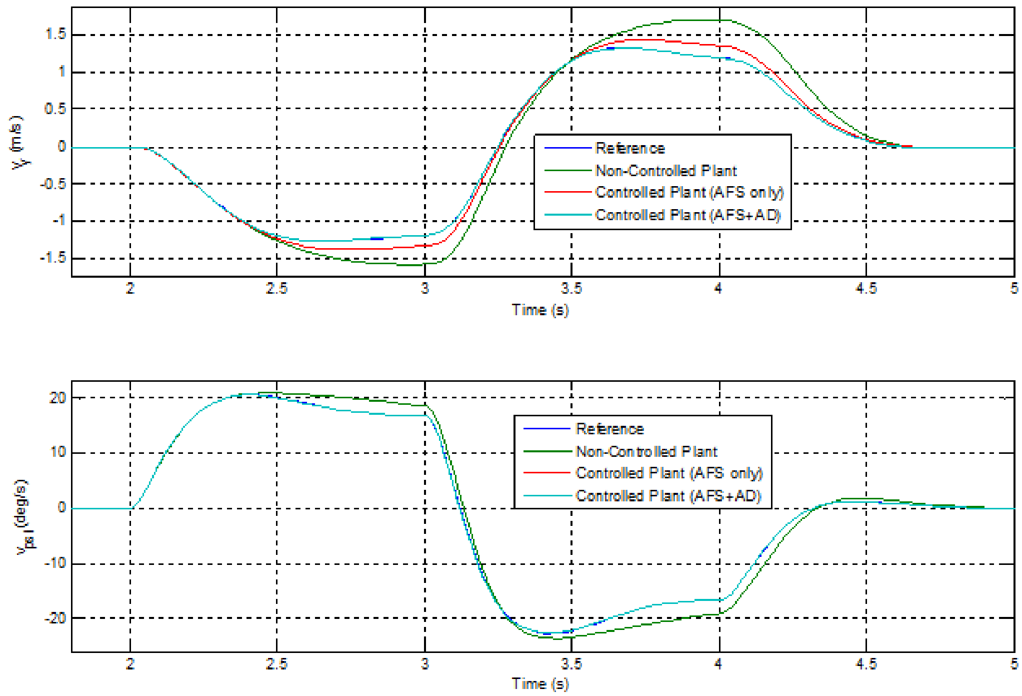
5.1.1 Double Step Steer

The first simulation is a double step steer manoeuvre, in which the vehicle velocity is set to 25 m/s (90 km/h). After 2 seconds (t=2), the drivers turns left (65°); in t=3, the steering wheel is turned right of the same angle. In t=4, the SWA goes back to zero.

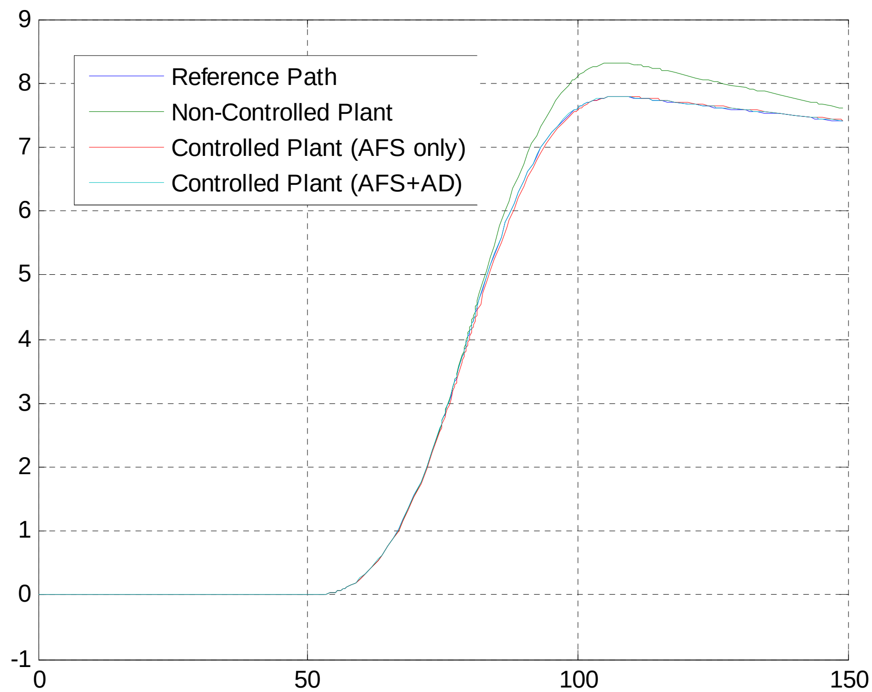


Steering Wheel Angle (SWA)

The simulations show that the MIMO has the best behavior, and the tracking of yaw rate and lateral velocity are almost perfect. The passive vehicle, in the second step steer, shows excessive values for the vehicle slip angle (and lateral velocity). Moreover the trajectory path is a bit larger than expected. This behavior is not comfortable for the driver and is corrected by the control systems, especially from the MIMO one. On the contrary, the tracking of the yaw rate is perfect in both the control solutions.

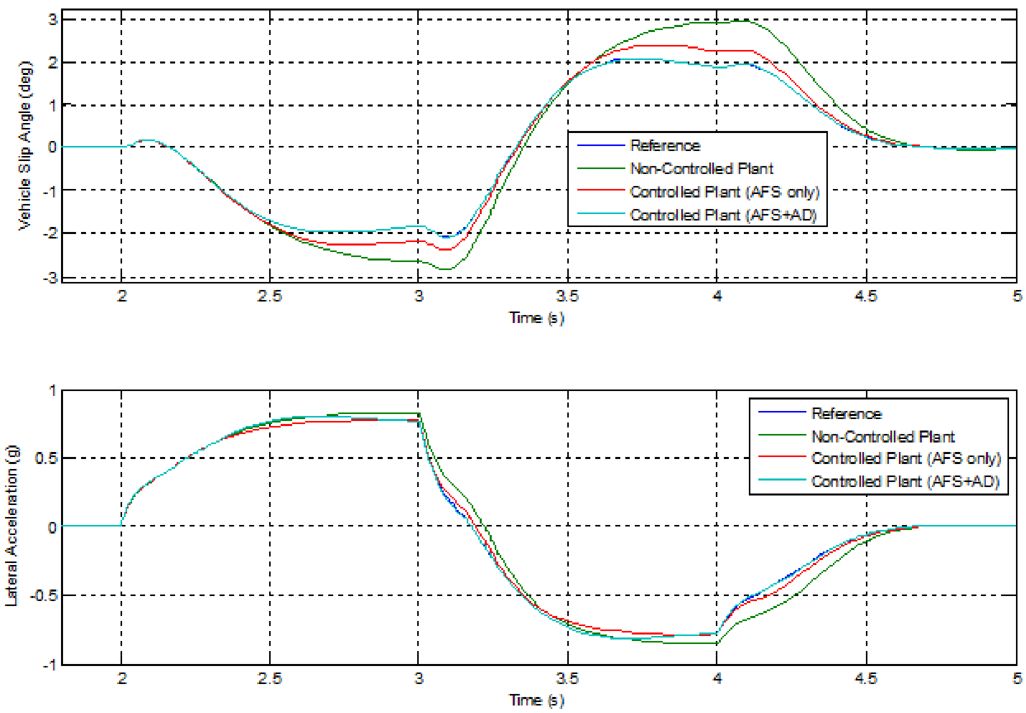


State variables



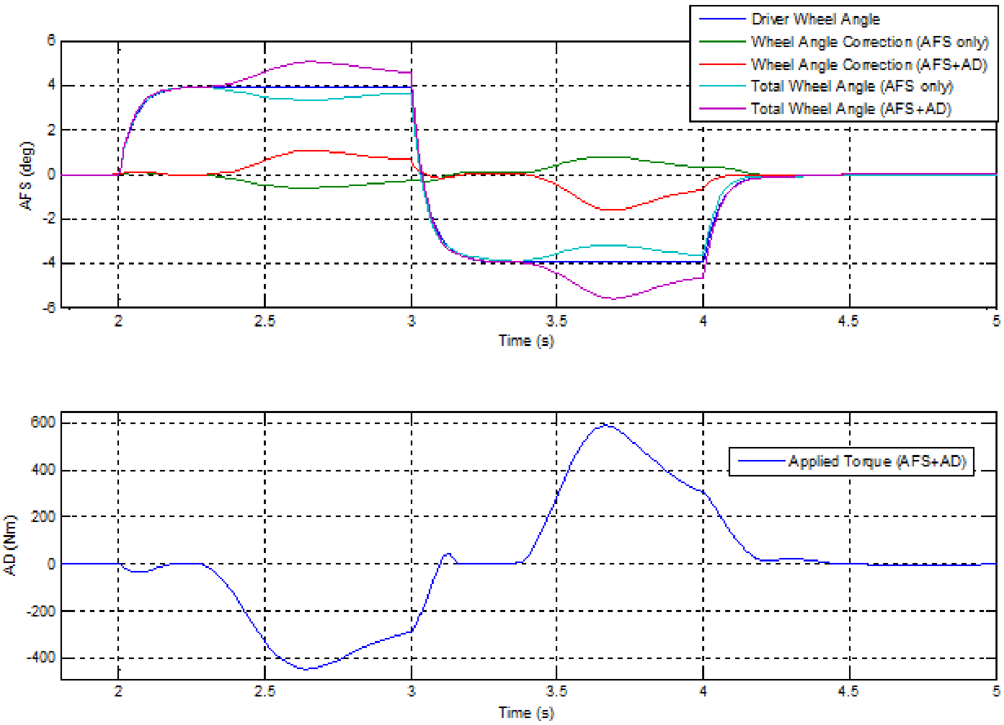
Trajectory

Values of about 0.8 g of lateral acceleration are reached in this manoeuvre.



Vehicle Slip Angle and Lateral Acceleration

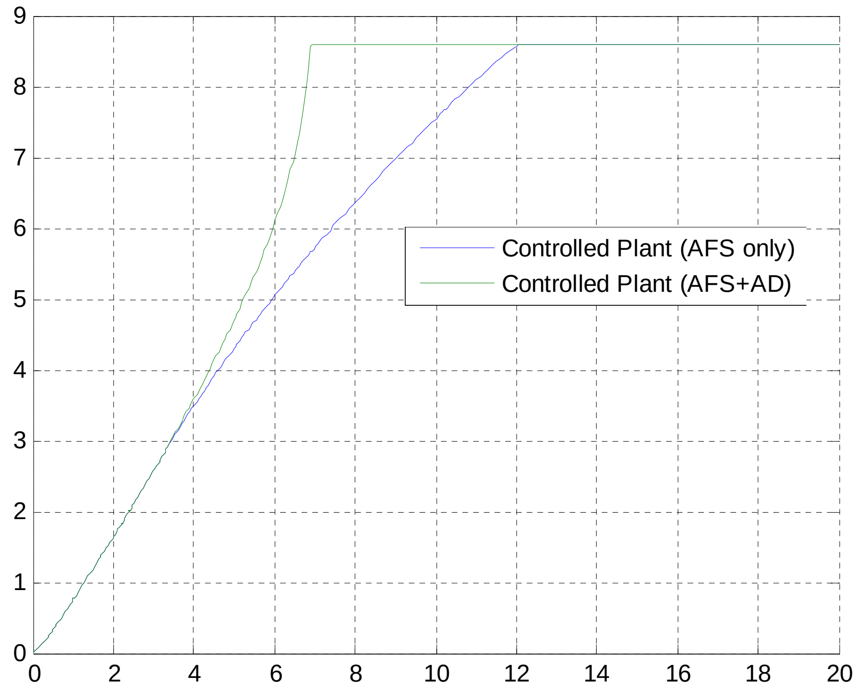
The action of the AFS, in the MIMO and in the SISO, are opposite. In fact, in the SISO control system, the actuator is only the AFS, so its action of tracking yaw rate is against the direction of steering. In the MIMO one, on the contrary, the AD action is against the direction of steering, while the AFS goes in the direction of the manouvre to balance the AD effect.



Inputs

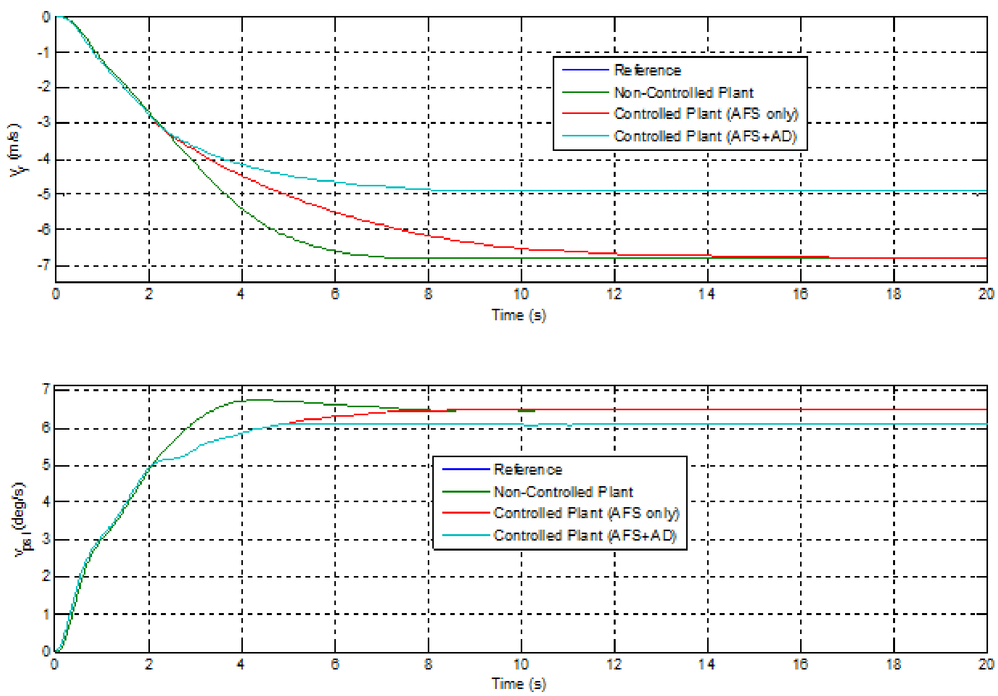
5.1.2 Ramp Steer

In the previous manouvre, the tracking was very good, because both of the actuators worked without limitations, since no applicability conditions were violated. To show the performance of tracking in saturation condition, we try a Ramp Steer manoeuvre. This simulation shows the problem which was described in the previous chapter for the integrated controller. In particular: the steering manoeuvre is in the anticlockwise direction. The MIMO, in order to track the lateral velocity reference, generates a negative torque (clockwise); consequently, the AFS action is reduced if compared to the AFS-only action, so the saturation of the front axle is reached before than in the AFS-only case.



Front Slip Angle

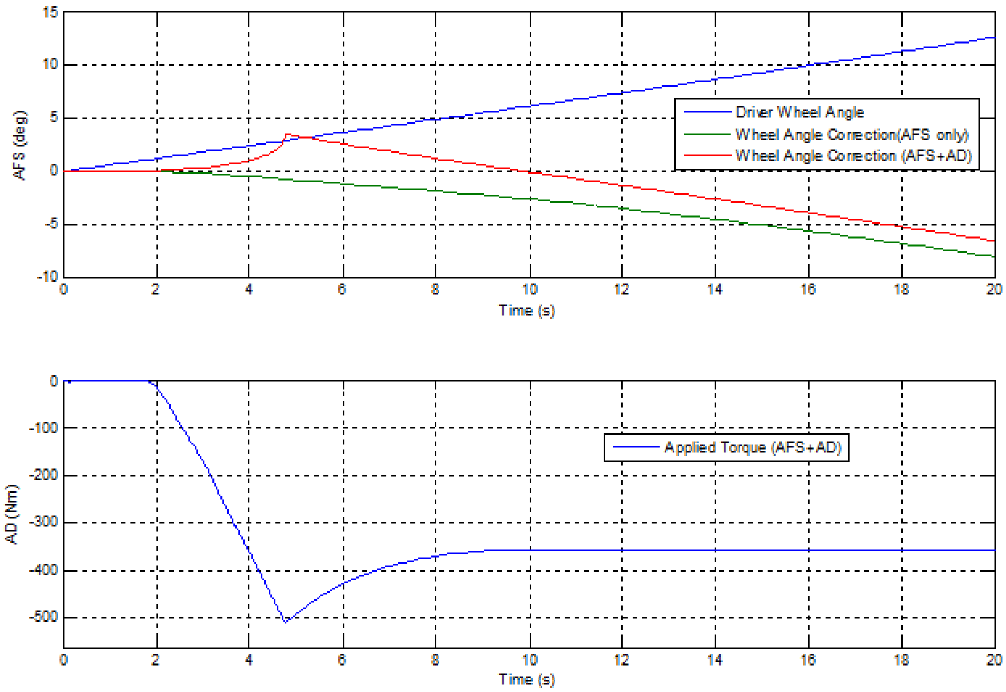
The AFS saturation prevents the MIMO controller to follow the desired yaw rate target, and the vehicle turns less than expected. On the contrary, the lateral velocity (and vehicle slip angle) are tracked very well, because the AD works without saturation.



State variables

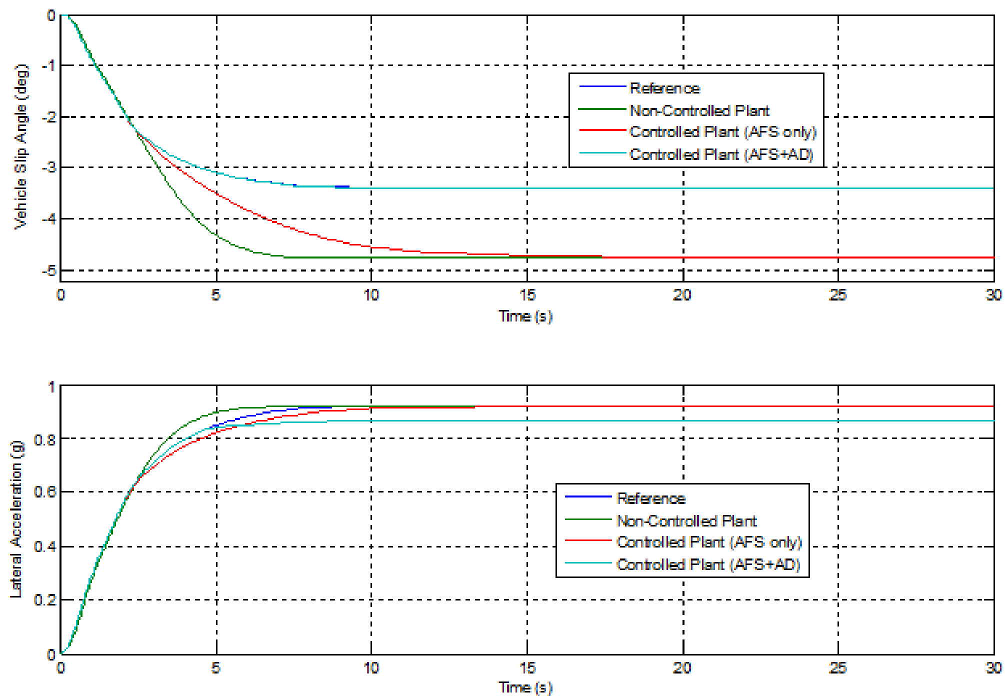
If no further assumptions are made on the AFS system, in saturation conditions it completely cancels the action of the driver, allowing the front axle to stop at the $\alpha_{f,sat}$ point

$$\delta_C = -\delta_D + \alpha_{f,sat} + \frac{v_y + l_f v_\psi}{v_x} \tag{4a}$$



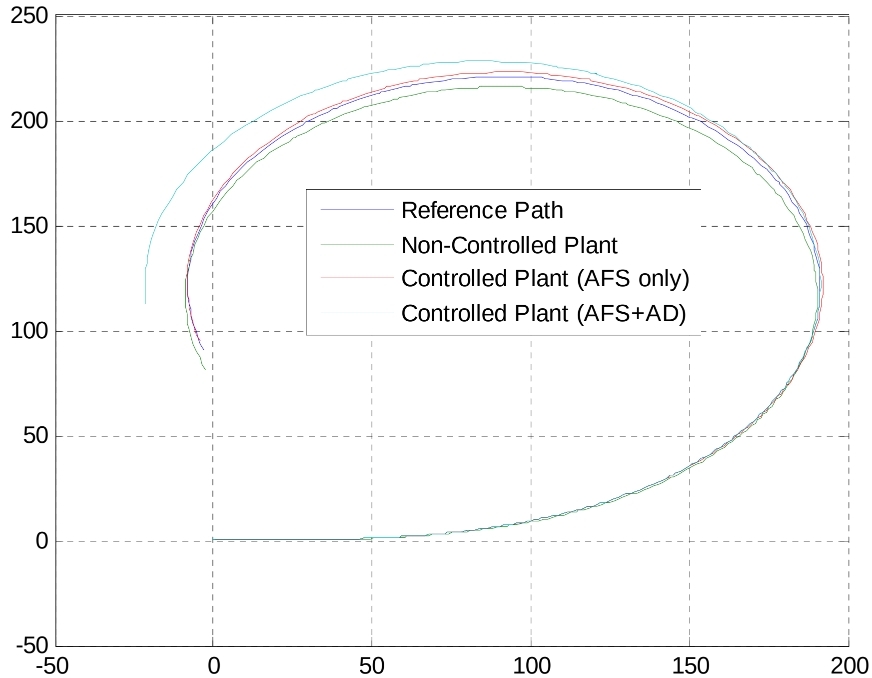
Inputs

Because of the high stiffness of the rear lateral characteristic (see Chapter 4, Reference), which causes the saturation of front axle before than expected, the maximum reachable value for the lateral acceleration is about $0.86g$ while the passive system and the AFS-only controlled system get to a value of about $0.91g$.



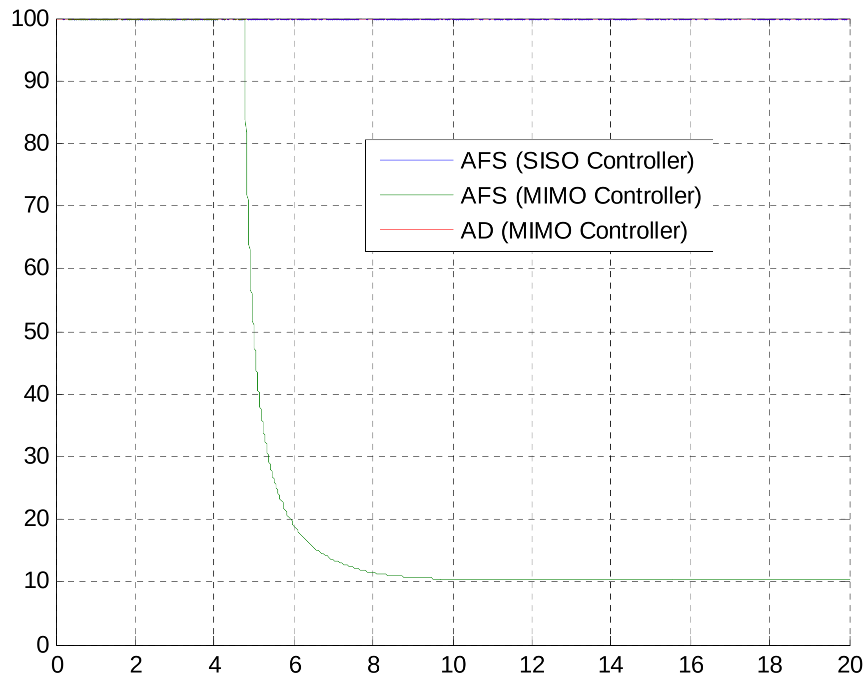
Vehicle Slip Angle and Lateral Acceleration

The MIMO controlled system turns less than expected and this can be easily observed looking at the trajectory path. This drawback can be overcome if the action of AD is limited by a dead zone or by a fixed gain. In these conditions, also in the MIMO controlled system a good yaw rate tracking can be achieved, and also higher values of lateral acceleration.



Trajectory

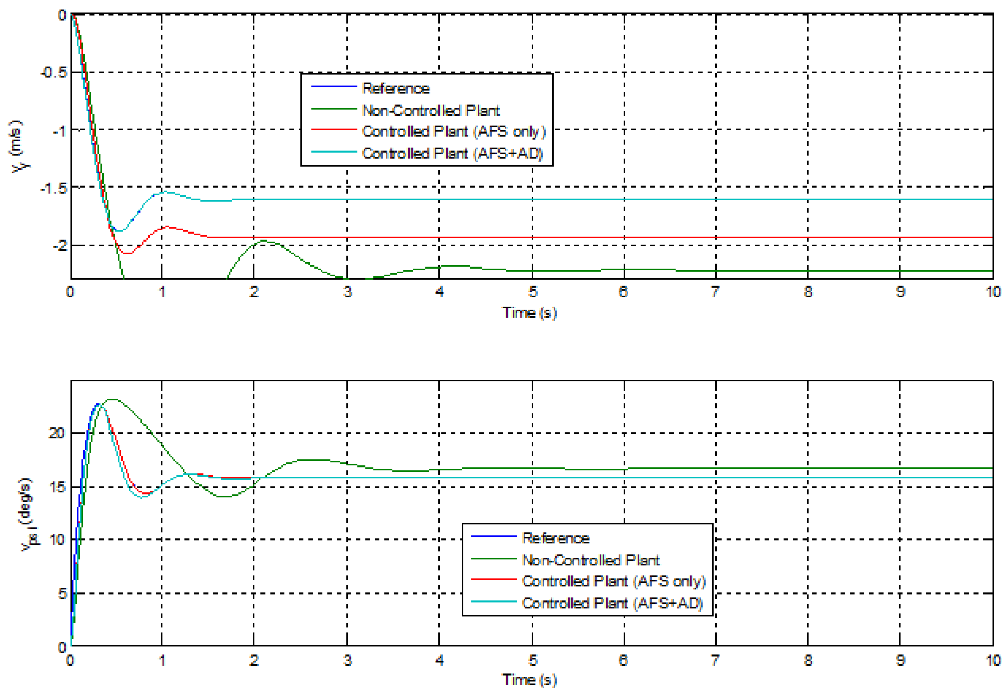
The saturation of AFS is showed also looking at the percentage of actuation, that rapidly decreases from 100% to 10%. It means that only a small part of the AFS calculated angle is really applied, and this causes the lack of yaw rate tracking.



Percentage of actuation

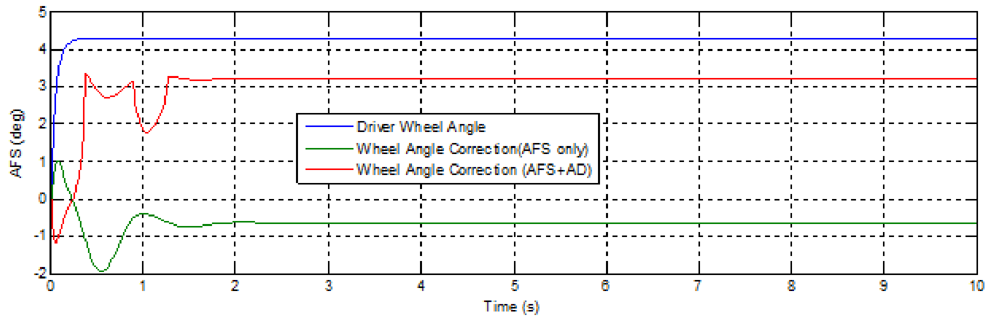
5.1.3 Constant radius turn

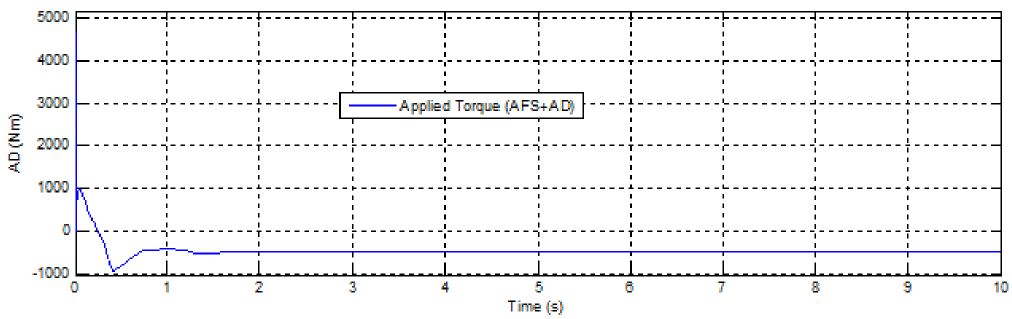
Similar considerations can be done, if we drive the vehicle in a constant radius turn. There is a light saturation of the front axle for the MIMO system, but a good matching for both the outputs is achieved. The oscillations of the passive vehicle in the lateral velocity are corrected by both of the controllers.



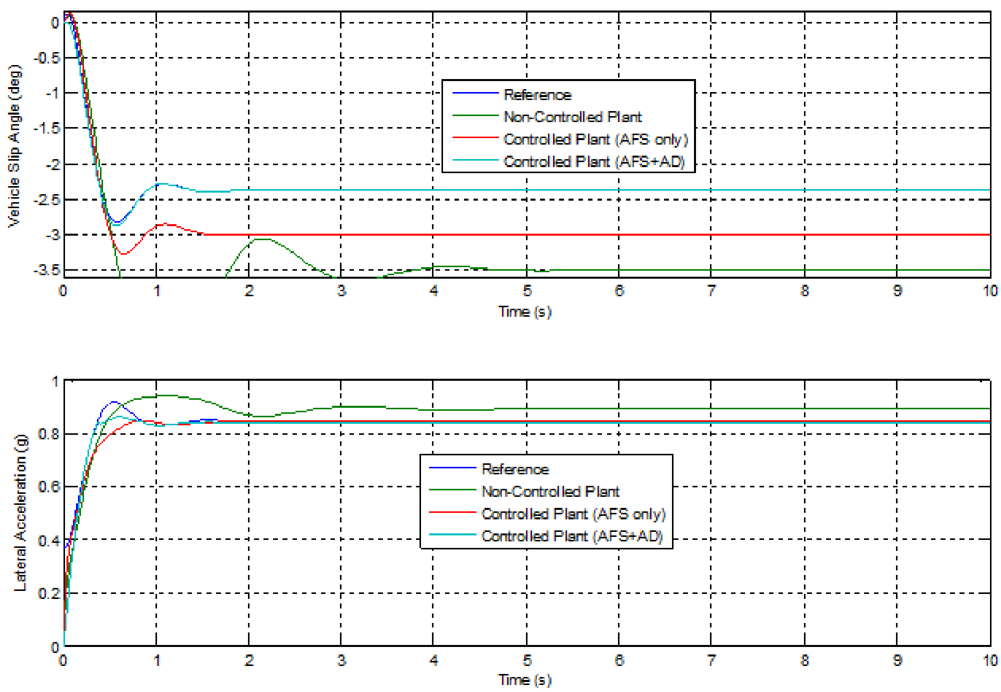
State variables

Once more, the opposite action of the AFS controllers can be pointed out. We can generally deduce that, while in the AFS-only control system the AFS correction is against the driver's SWA direction, in the MIMO system, because of the possibility of share the control action on 2 actuators, the AFS correction has usually a lower value, and it can also be zero or in the direction of the steering, according to the differential influence on the yaw rate dynamics.

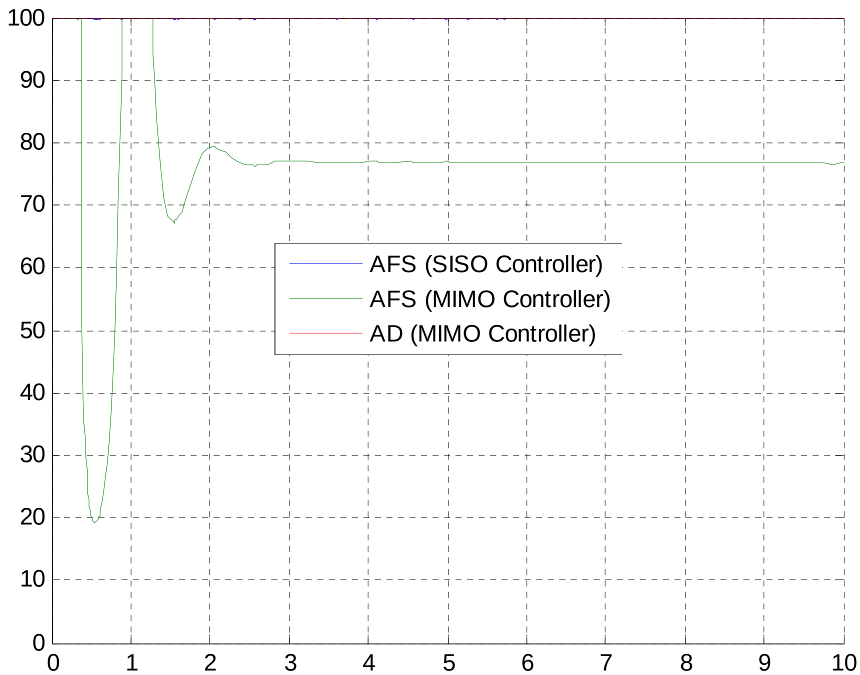

AFS (deg)
Time (s)
Driver Wheel Angle
Wheel Angle Correction(AFS only)
Wheel Angle Correction (AFS+AD)


AD (Nm)
Time (s)
Applied Torque (AFS+AD)

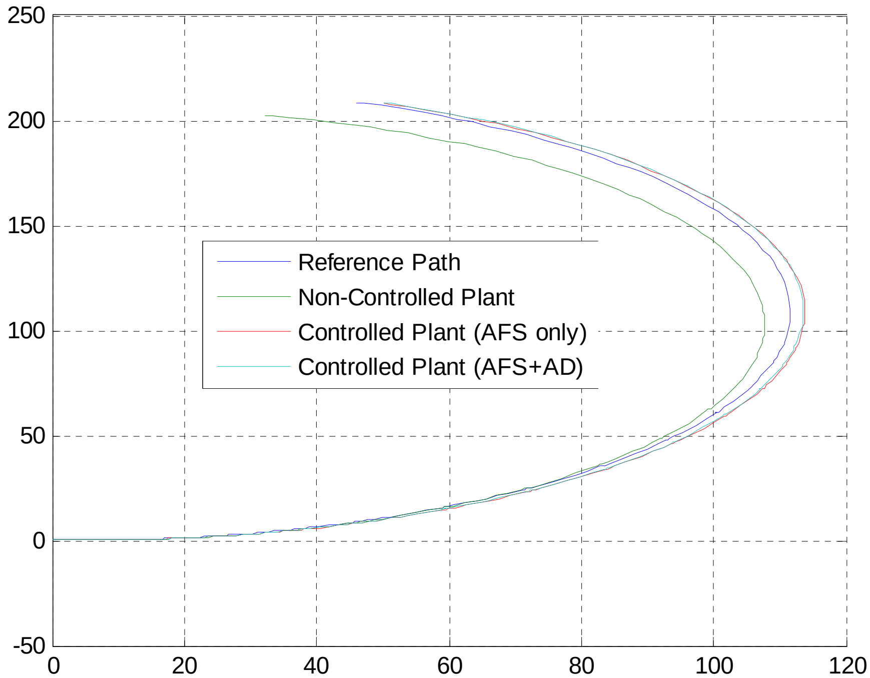
Inputs



Vehicle Slip Angle and Lateral Acceleration



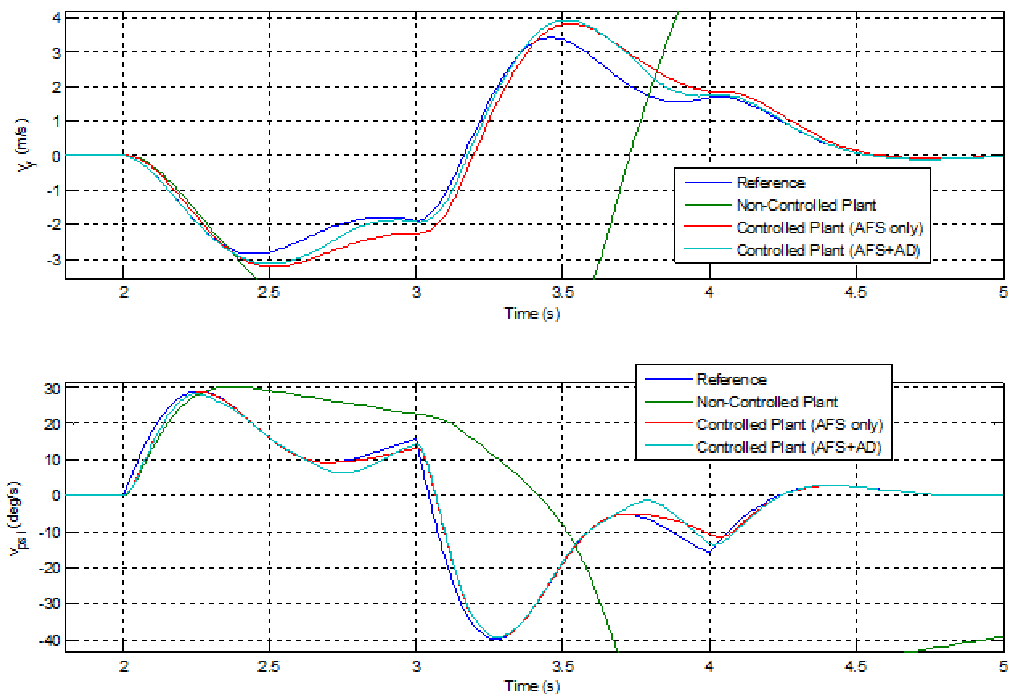
Percentage of Actuation



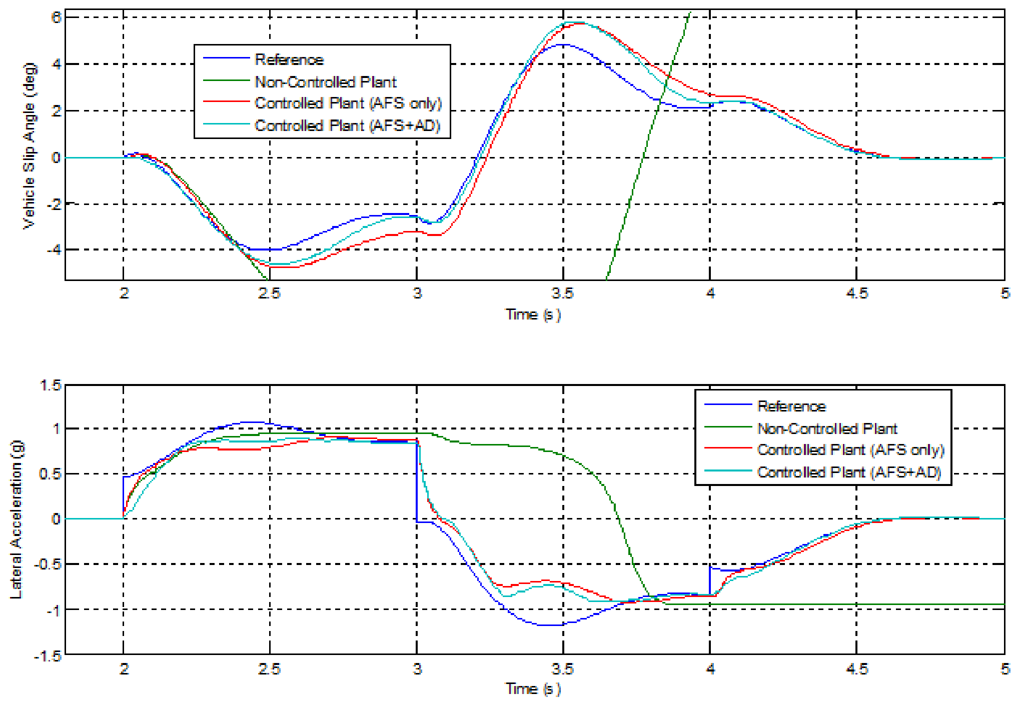
Trajectory

5.1.4 Double Step Steer (in saturation conditions)

Now, we describe a hard manoeuvre, in which also the SISO AFS-only system reaches saturation conditions. We consider a double step steering manoeuvre, with vehicle speed of 35 m/s (126 km/h), with step steers of $\pm 120°$. In these conditions, values of more than $0.9g$ of lateral acceleration are reached by all the considered systems. The non-controlled plant goes unstable during the second step of the steering manoeuvre. Both the controllers keep the vehicle stable, even reaching the same lateral acceleration of the passive vehicle. The output tracking is not perfect because of the saturation of the actuators, but usually the MIMO controller has a better tracking of the lateral velocity, while in the yaw rate tracking, the level of the performance is the same.

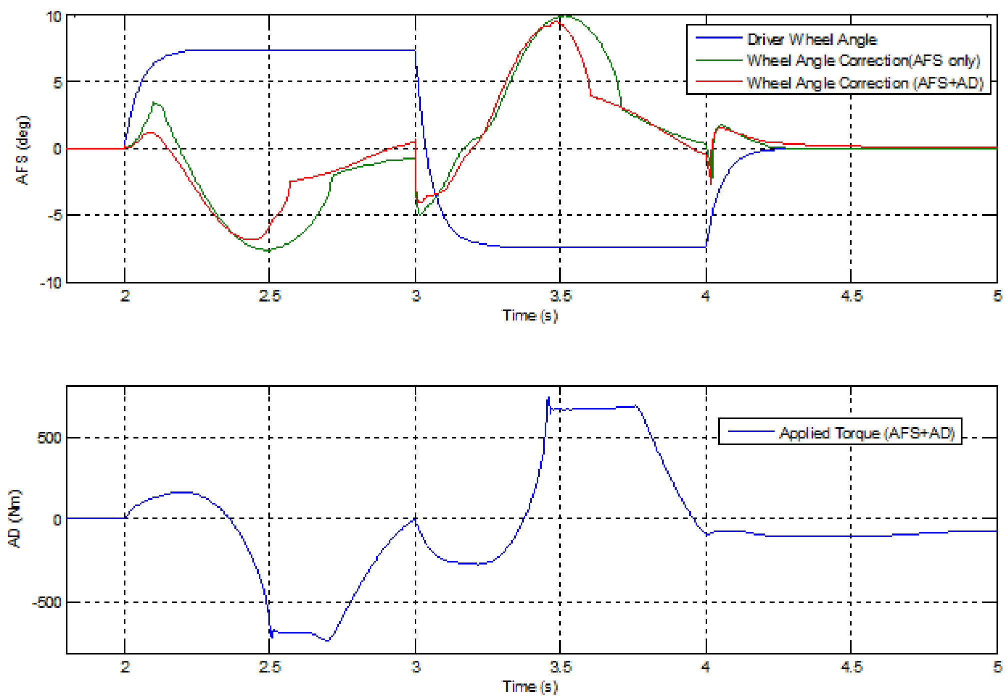


State variables



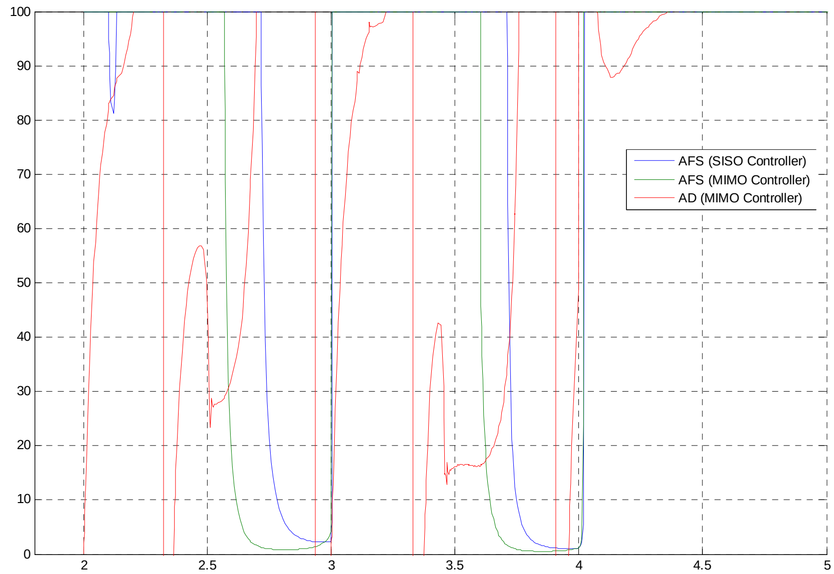
Vehicle Slip Angle and Lateral Acceleration

The behavior of the actuators is quite intuitive and the considerations are quite similar to the previous ones, especially for the AD action (always against the SWA direction) and for the lower angle correction values of the AFS, in the combined solution, if compared to the AFS-only controller.



Inputs

The plot of the percentage of actuation shows that each of the actuator is limited during the tracking control. Usually the MIMO periods of front saturation are longer than the AFS-only ones. The AD saturates as well, because the combination of (α_r, k_r) goes to values near to the non-invertibility condition (see Chapter 4).



Percentage of actuation

The trajectory plot shows a better behavior for the MIMO system, if compared to the AFS-only one.

5.2 Non-nominal Environment

5.2.1 Control in presence of strong wind conditions

In the following simulations, the control systems are tried in non-nominal conditions. With the perturbations given by the wind, the linearization of the system is no more perfect, so nothing can be guaranteed. Actually, the controllers keep the vehicle stable, and the MIMO control system shows its best performance.

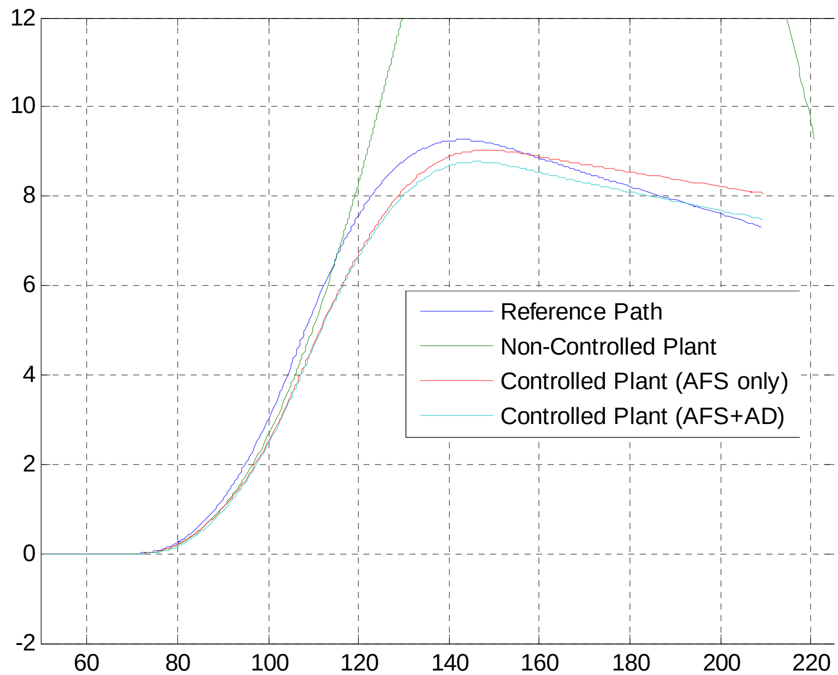
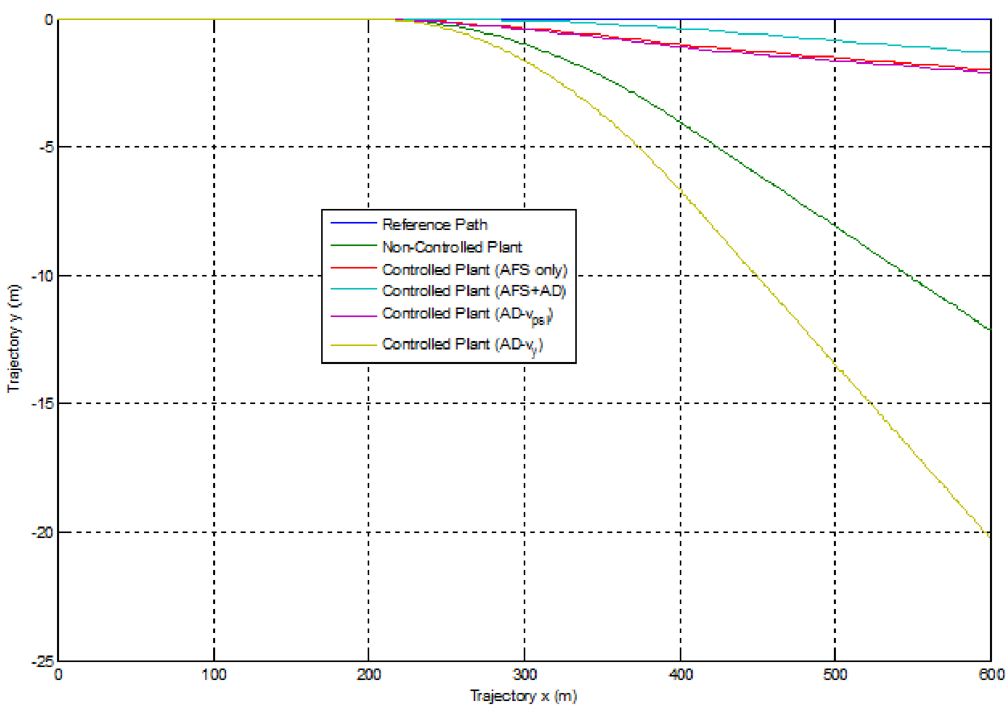


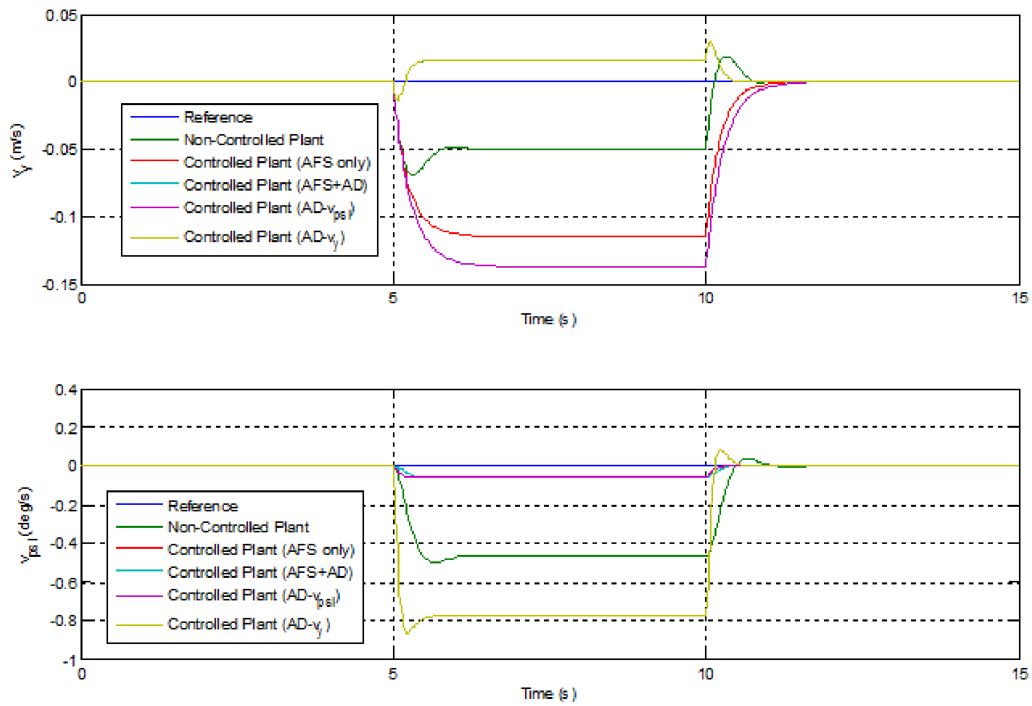
Figure 5.1: Trajectory

We consider a strong wind impulse (40 m/s, 1000 N, for 5 seconds), while the vehicle is going forward, with SWA=0. Without considering the reaction of the driver, the trajectory plot shows that the non-controlled vehicle goes far from the path, while the MIMO controller has the best behavior. The effect of the wind is quite reduced, and the driver has time to react and steer against the drift direction. In the following plots, the performance of the AD-only control systems (see Chapter 3) are considered as well.

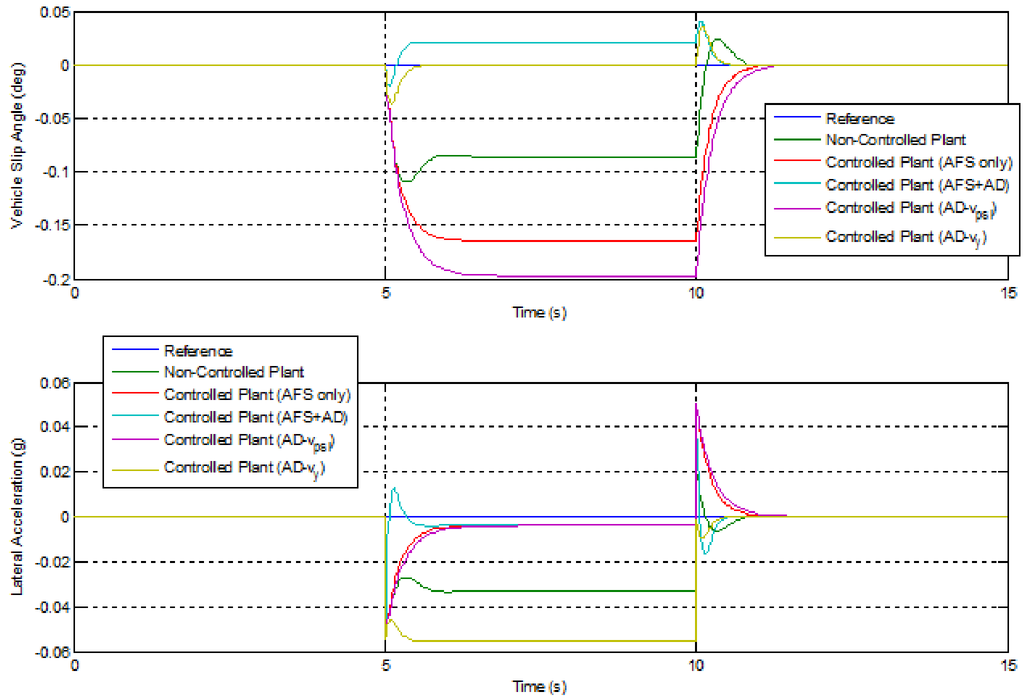
The better behavior, for MIMO control system, is achieved thanks to the control of the vehicle slip angle, not present in the SISO (AFS-only) solution.



Trajectory

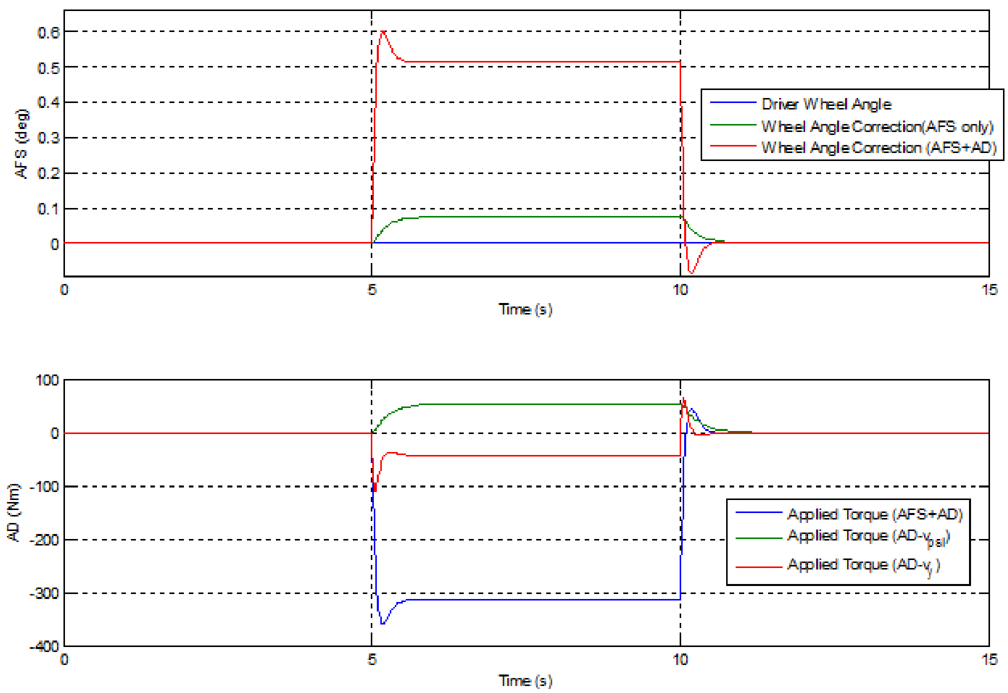


State variables



Vehicle Slip Angle and Lateral Acceleration

Considerations about the actuations are similar to those ones in the previous simulations.

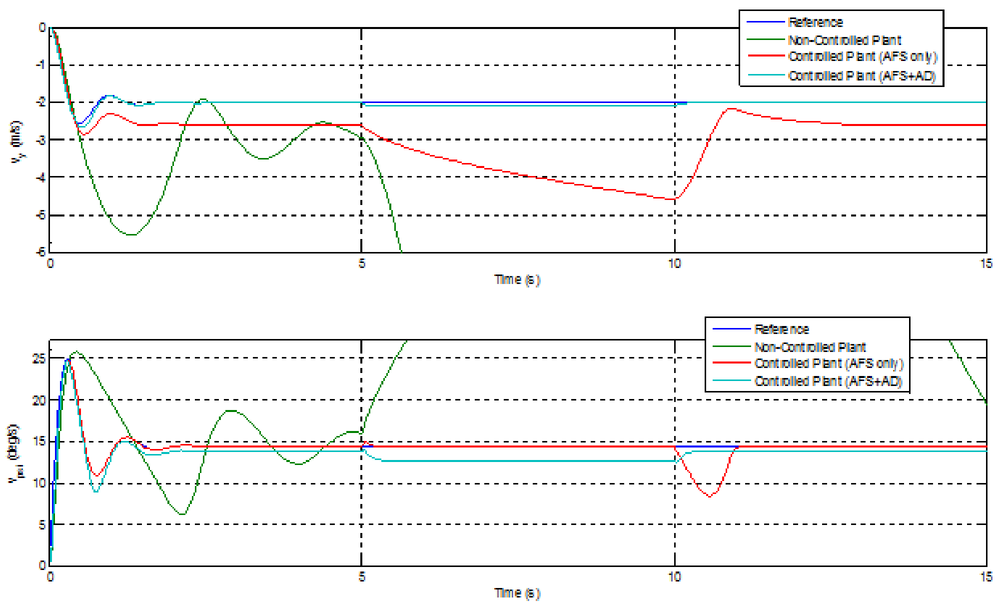


Inputs

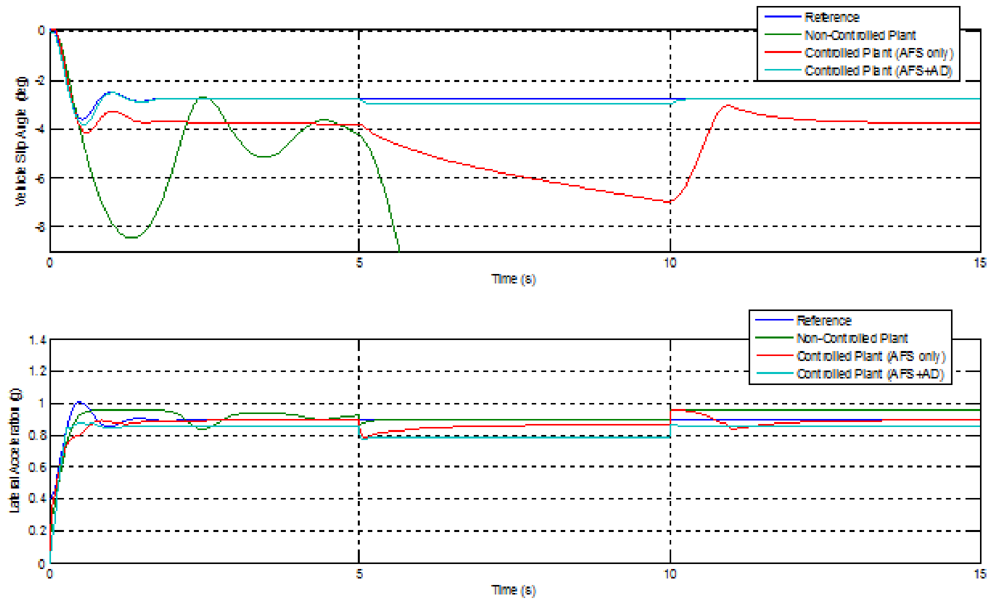
The same perturbation can be observed during a constant radius turn manoeuvre (40 m/s, 144 km/h, fixed SWA=100°).

Of course, the discontinuity of the control action is caused by the wind impulse, but its behavior can be explained as in the previous cases.

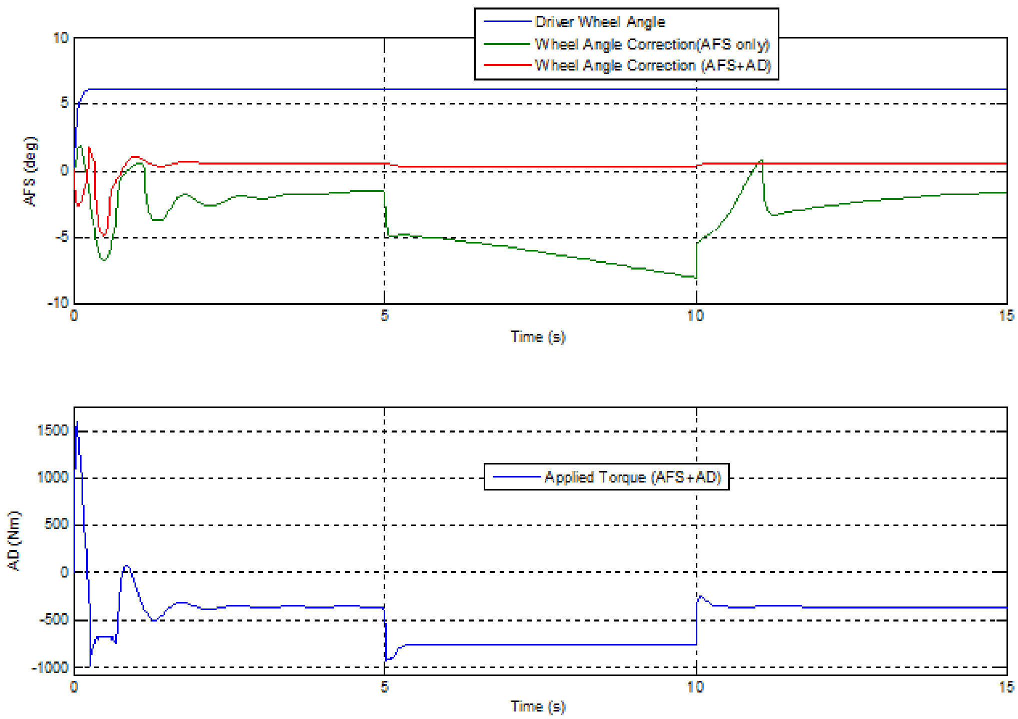
As usual, the MIMO tracking of lateral velocity is very good. On the contrary, in the SISO controlled plant, the lack of control on the vehicle slip angle slip would drive the vehicle to instability, if the duration of the impulse were longer than 5 seconds. The yaw rate tracking is quite satisfying in both the considered controllers.



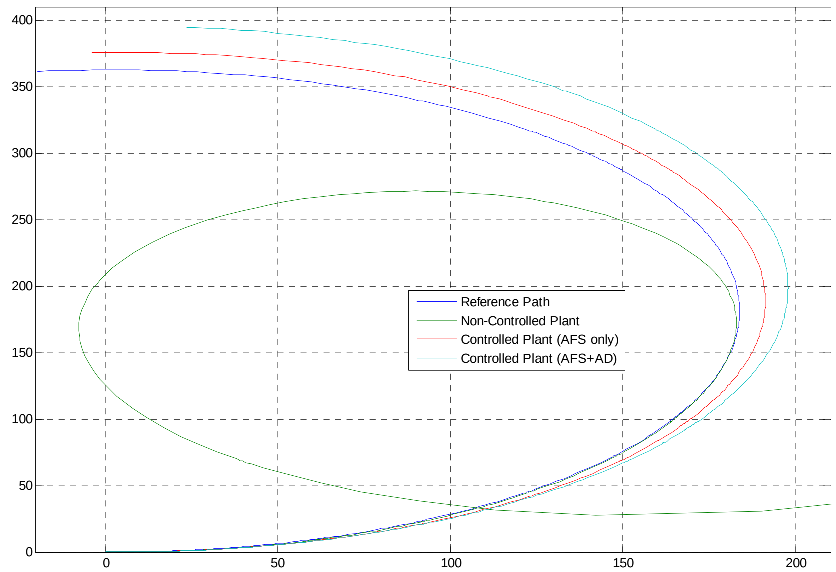
State variables



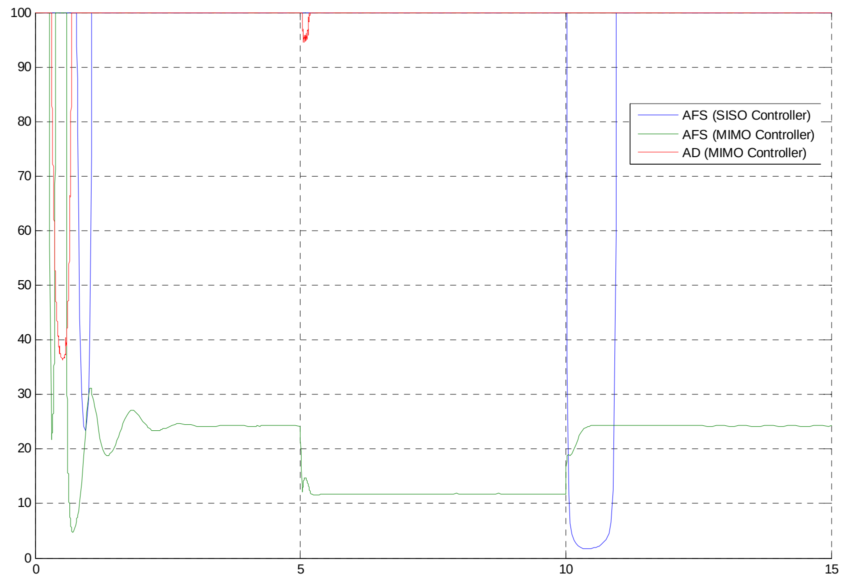
Vehicle Slip Angle and Lateral Acceleration


Driver Wheel Angle
Wheel Angle Correction(AFS only)
Wheel Angle Correction (AFS+AD)
AFS (deg)
Time (s)
Applied Torque (AFS+AD)
AD (Nm)
Time (s)

Inputs


400
350
300
250
200
150
100
50
0
0
50
100
150
200
Reference Path
Non-Controlled Plant
Controlled Plant (AFS only)
Controlled Plant (AFS+AD)

Trajectory



Percentage of actuation

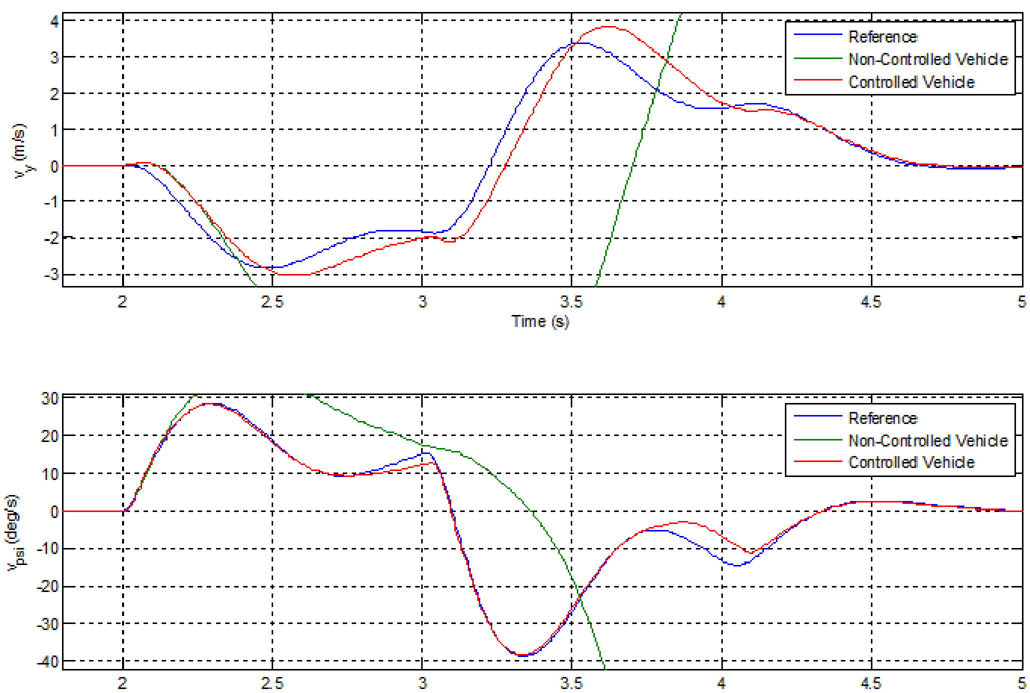
5.2.2 Control of the real vehicle (Black Box)

Simulations on the real vehicle (Black Box Model) are available only for the SISO (AFS-only) control system, at the moment. In fact, this controller has a much simpler behavior than the other ones:

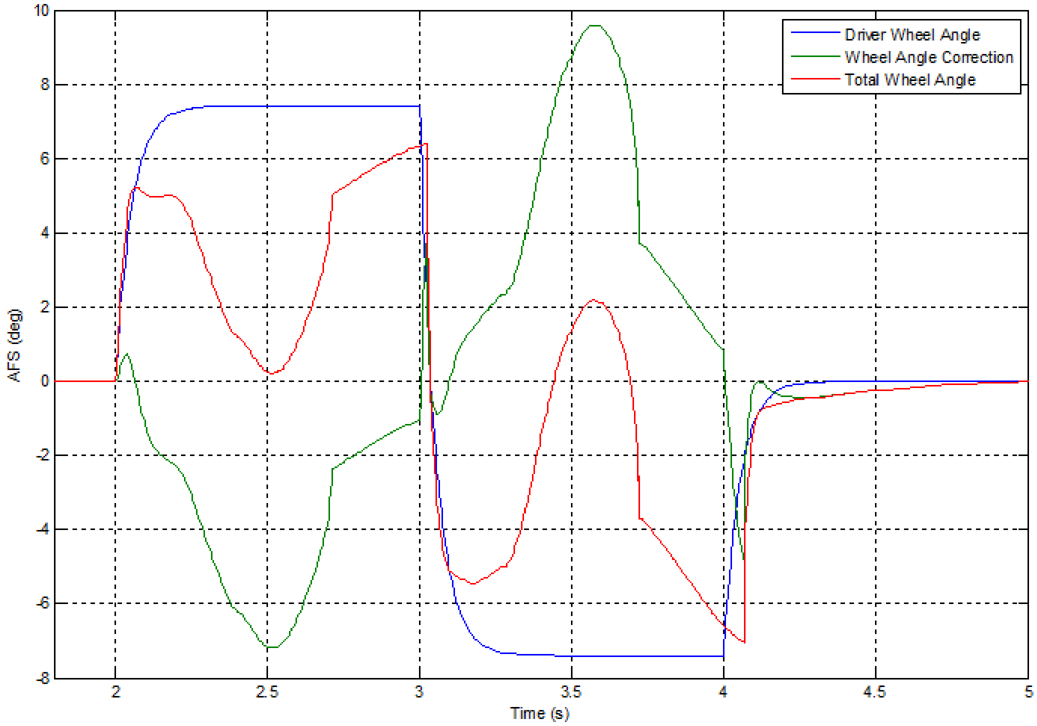
- it involves only the tyres lateral characteristics
- it is less sensible to the parameter uncertainty or to unmodeled dynamics because there is less computation effort
- less measurements are needed on the vehicle, and so there is less sensitivity to possible errors

- the other controllers work in combined slip conditions, and in this case the identified characteristics are not optimally fit (compromise between accuracy and simplicity)

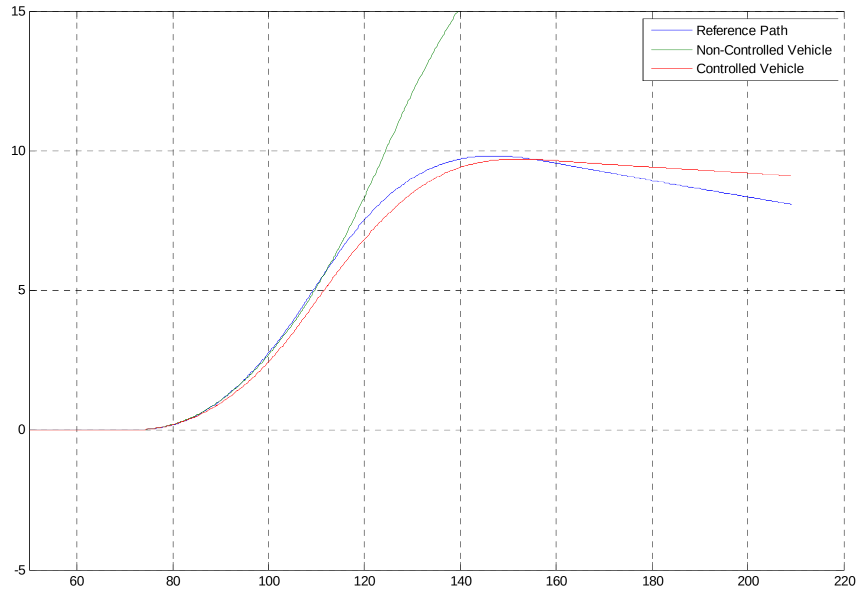
The manoeuvre (double step steer, 35 m/s, 126 km/h, ±120° SWA) shows that the SISO controller achieves a very good tracking of the yaw rate and of the trajectory, while the non-controlled vehicle goes unstable in the second step of the steering manoeuvre.



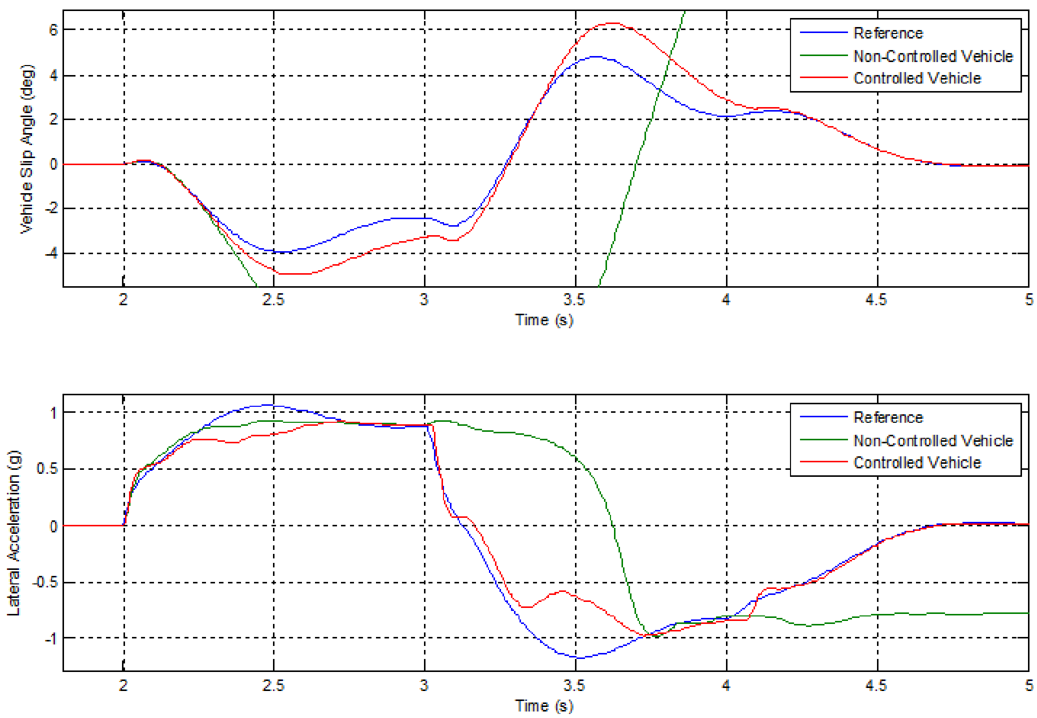
State variables


Driver Wheel Angle
Wheel Angle Correction
Total Wheel Angle
AFS (deg)
Time (s)
10
8
6
4
2
0
-2
-4
-6
-8
2
2.5
3
3.5
4
4.5
5

Inputs


15
10
5
0
-5
60
80
100
120
140
160
180
200
220
Reference Path
Non-Controlled Vehicle
Controlled Vehicle

Trajectory



Vehicle Slip Angle and Lateral Acceleration

Chapter 6

Conclusions

6.1 Results

This work wants to evaluate the possibility of integrating two actuators in order to solve a typical control problem. The actuators are the Active Front Steer (AFS) and the Active Differential (AD); they have been described in the previous chapters. The control problem is a tracking control problem. The outputs to track are yaw rate and lateral velocity.

The work is made up of 3 parts: modeling, identification and control.

In the modelling phase, the starting point was a 6 dof vehicle model; after many simplifying assumptions, we got a modified bicycle model, with only two degrees of freedom. However, it is more complex than a single track model since it takes care of the non-linear behavior of the tyres and of the combined presence of longitudinal and lateral slip.

In the identification phase, an optimum matching between plant model and Black Box model with respect to the situation of pure lateral/longitudinal slip was achieved. For the situation of combined slip conditions, we considered a reasonable compromise between accuracy and simplicity (computational effort) in the estimate of tyre characteristics. These functions were fitted in two steps. At first, the sample functions were built by means of many measurements on the vehicle model, in steady-state conditions. Then, analytic functions have been fitted to approximate the sample functions.

Finally, in the control phase, several controllers were designed and implemented, with different features and goals. The control technique was always the Feedback Linerization Method. The advantage of this technique is that it solves the control problem in a closed form, differently from other methods of control design.

6.2 Control features

All the controllers have an optimal asymptotic tracking behavior, in nominal conditions. However, only the MIMO controller is able to track both the target functions, and so it has some advantages. In saturation, in spite of a worse performance in tracking, at least the stability of the system is always guaranteed.

The MIMO solutions have some advantages if compared to the SISO ones. In fact the simultaneous tracking has the consequence that the vehicle trajectory is tracked as well. Moreover, lower values of vehicle slip angles are measured in the MIMO controlled vehicles, so the driver comfort is improved. Finally, a better robustness is showed by the integrated controller in conditions of side wind. A further advantage for the MIMO is that the control action is shared by two actuators. So, in conditions of constant saturations on the actuators, the limitation condition is reached later than in the SISO systems.

The MIMO control system has some drawbacks as well. In particular, a higher computational effort is needed to achieve the good results of the combined control, and more measurements and sensors are involved. Moreover, some simulations show that the AD action drives sometimes the front axle till its saturation point.

6.3 Next steps

This work is the first step towards the implementation of an integrated controller AFS+AD. Possible modifications and extensions of this work can follow different directions.

In the modelling, for instance, some assumptions can be removed, to get to more complex and realistic models, achieving a better matching with the vehicle behavior. Examples of assumptions that can be removed are the following ones:

- constant vertical tyre loads
- free rolling torque disregarded (set equal to zero)

Moreover the tyre forces can be characterized in a different manner, for example by brush models instead of Magic Formulas.

In the identification phase, alternative methods of identification, for example taking the transient response of the system into account, can be used for a more realistic behavior of the identified model, during the dynamic manoeuvres. In this way, other parameters, such as moments of inertia, can be estimated. Moreover, the tyre behavior can be characterized in a better way by two-variable tyre characteristics, removing the hypothesis described in section 2.4, with its drawbacks.

With regard to the control problem, a better performance of the feedback linearization, in the MIMO case, can be consequence of a 2-dimensional characterization of tyres. Moreover, alternative methods of control can be considered as well, less simple but with better robustness if compared to the feedback linearization.

The implementation of the controller can be modified by means of different techniques of management of saturation conditions; improvements can be achieved as well with a more suitable generation of the target functions, in order to reduce unwanted actuations.

The goal of improving stability can be reached also letting the MIMO controller run in parallel with a brakes controller; a more complex solution is to design an integrated controller AFS+AD+brakes, evaluating stability improvements.

Appendix A

Least Square Method

Least squares analysis (see [10]), also known as ordinary least squares analysis, is a method that determines the values of unknown quantities in a model by minimizing the sum of the residuals (difference between the predicted and observed values) squared. This method was first described by Carl Friedrich Gauss at the turn of the 19th century. Today, this method is available in most statistical packages. Furthermore, many other types of optimization problems can be expressed in a least squares form, by either minimizing energy or maximizing entropy.

A.1 Problem statement

Suppose that the data set consists of the points (x_i, y_i) with $i = 1, 2, ..., n$. We want to find a function $f(p, x)$ such that $f(p, x_i) \approx y_i$, where p is a vector containing the parameters for the given model. For example, if a quadratic function given by $f(x) = ax^2 + bx + c$ is to be used, then the parameter vector p would be given by

$$p = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

In least squares analysis, the objective is to minimize the following function

$$S = \sum_{i=1}^{n} \left(y_i - f\left(p, x_i\right)\right)^2$$

subject to the parameter vector. The above minimization explains the origin of the name least squares.

A.2 Solving the least squares problem

In general, since this is an unconstrained general optimization problem, there is no solution to the least-squares problem. General methods, such as Newton's method combined with the gradient descent method, or specialized methods for least squares analysis, such as the Gauss-Newton algorithm or the Levenberg-Marquardt algorithm, can be used.

For cases, where the problem is linear in the parameters, then an exact solution can be obtained. A model is said to be linear if the first-order, partial derivatives of the function with respect to the parameters is independent of the parameters themselves. Under certain circumstances, a function can be linearized by re-arranging the equation.

A.3 Least squares and regression analysis

In regression analysis, one replaces the relation

$$f(x_i) \approx y_i$$

by

$$f(x_i) = y_i + \epsilon_i$$

where the noise term ϵ is a *random variable* with mean zero. Note that we are assuming that the x values are exact, and all the errors are in the y values.

Again, we distinguish between linear regression, in which case the function f is linear in the parameters to be determined (e.g., $f(x) = ax^2 + bx + c$), and nonlinear regression. As before, linear regression is much simpler than nonlinear regression. It is tempting to think that the reason for the name linear regression is that the graph of the function $f(x) = ax + b$ is a line. Fitting a curve $f(x) = ax^2 + bx + c$, estimating a, b, and c by least squares, is an instance of linear regression because the vector of least-square estimates of a, b, and c is a linear transformation of the vector whose components are $f(x_i) + \epsilon_i$.

A.4 Parameter estimates

By recognizing that the regression model is a system of linear equations we can express the model using data matrix X, target vector Y and parameter vector δ. The i-th row of X and Y will contain the x and y value for the i-th data sample. Then the model can be written as

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix}$$

which when using pure matrix notation becomes

$$Y = X\ \delta + \epsilon$$

where ϵ is normally distributed with expected value 0.

The least-squares estimator for δ is

$$\hat{\delta} = \left(X^T X\right)^{-1} X^T Y$$

and the sum of squares of residuals is

$$Y^T \left(I_n - X \left(X^T X \right)^{-1} X^T \right) Y$$

Appendix B

Feedback linearization method

This chapter provides a description of feedback linearization (see [3] for more informations), including what it is, how to use it for control design and what its limitations are.

B.1 Feedback Linearization and the Canonical Form

In its simplest form, feedback linearization amounts to canceling the nonlinearities in a nonlinear system so that the closed-loop dynamics is in a linear form. The idea of canceling the nonlinearities and imposing a desired linear dynamics can be applied to a class of nonlinear systems described by the so-called *companion form*, or *controllability canonical form*.

Definition 1 *A system is said to be in companion form if its dynamics is represented by*

$$x^{(n)} = f(x) + g(x)u \tag{B.1}$$

where u is the scalar control input, x is the scalar output of interest, $x = [x, \dot{x}, ..., x^{(n-1)}]^T$ is the state vector, and f(x) and g(x) are nonlinear functions of the states.

This form is unique in fact that, although derivatives of x appear in this equation, no derivative of the input u is present. Note that, in state-space representation, (A.1) can be written

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} x_2 \\ \vdots \\ x_n \\ f(x) + g(x)u \end{bmatrix}$$

For systems which can be expressed in the controllability canonical form, using the control input (assuming g to be non-zero)

$$u = \frac{1}{g}[v - f]$$

we can cancel the nonlinearities and obtain the simple input-output relation (multiple-integrator form)

$$x^{(n)} = v$$

Thus, the control law

$$v = -k_0 x - k_1 \dot{x} - ... - k_{n-1} x^{(n-1)}$$

with the k_i chosen so that the polynomial $p^n + k_{n-1}p^{n-1} + ... + k_0$ has all its roots strictly in the left-half complex plane, leads to the exponentially stable dynamics

$$x^{(n)} + k_{n-1}x^{(n-1)} + ... + k_0 = 0$$

which implies that $x(t) \longrightarrow 0$. For tasks involving the tracking of a desired output $x_d(t)$, the control law

$$v = x_d^{(n)} + k_0 e + k_1 \dot{e} + ... + k_{n-1} e^{(n-1)}$$

(where $e(t) = x_d(t) - x(t)$ is the tracking error) leads to exponentially convergent tracking, Note that similar results would be obtained if the scalar x was replaced by a vector and the scalar g by an invertible square matrix.

Where the nonlinear dynamics is not in a controllability canonical form, one may have to use algebraic *transformations* to first put the dynamics into the controllability form before using the above feedback linearization design, or to rely on partial linearization of the original dynamics, instead of full linearization. These are the topics of the next sections.

B.2 Input-State Linearization and Input-Output Linearization

B.2.1 Input-State Linearization

Consider the problem of designing the control input u for a single-output non linear system of the form

$$\dot{x} = f(x, u)$$

The technique of input-state linearization solves this problems in two steps. First, one finds a state transformation $z = z(x)$ and an input transformation $u = u(x, v)$ so that the nonlinear system dynamics is transformed into an equivalent *linear time-invariant* dynamics, in the familiar form $\dot{z} = Az + bv$. Second, one uses standard linear techniques to design v.

B.2.2 Input-Output Linearization

Let us now consider a tracking control problem. Consider the system

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x) \end{aligned}$$

and assume that our objective is to make the output $y(t)$ track a desired trajectory $y_d(t)$ while keeping the whole state bounded, where $y_d(t)$ and its time derivatives up to a sufficiently high order are assumed to be known and bounded. The difficulty of the tracking control design can be reduced if we can *find a direct and simple relation between the system output y and the control input u*. Indeed, this idea constitutes the intuitive basis for the so-called input-output linearization approach to nonlinear control design.

B.2.3 Relative degree

If we need to differentiate the output of a system r times to generate an explicit relationship between the output y and the input u, the system is said to have *relative degree r*. As we shall see later, for any controllable system of order n, it will take *at most n* differentiations of any output for the control input to appear, i.e. $r \leq n$. This can be understood intuitively: if it took more than n differentiations, the system would be of order higher than n; if the control input never appeared, the system would not be controllable.

B.2.4 Internal dynamics

If $r < n$, it means that a part of the system dynamics, has been rendered unobservable in the input-output linearization. This part of the dynamics will be called the internal dynamics, because it cannot be seen from the external input-output relationship. If this internal dynamics is stable, our tracking control design problem can indeed be solved. Otherwise, the tracking controller is practically meaningless, because the instability of the internal dynamics would imply undesirable phenomena. Therefore, the effectiveness of the control design hinges upon the stability of the internal dynamics.

If $r = n$, there is no internal dynamics associated with this input-output linearization. Thus, in this case, input-output linearization leads to input-state linearization, and output tracking can be easily achieved.

B.2.5 Zero-dynamics

It can be shown that, in linear systems, the stability of the internal dynamics is simply determined by the location of the zeros. Extending the notions of zeros to non linear systems is not trivial. A way to approach these difficulties is to define a so-called *zero-dynamics* for a nonlinear system. The zero dynamics is defined to be the internal dynamics of the system when the system output is kept at zero by the input. The zero-dynamics is an intrinsic property of a nonlinear system, because the specification of maintaining the system output at zero uniquely defines the required input.

The reason for defining and studying the zero-dynamics is that we want to find a simpler way of determining the stability of the internal dynamics. In fact, the zero-dynamics only involve the stability of internal states, while the internal dynamics is coupled to the external dynamics and desired trajectories. For stabilization problems, it can be shown that local asymptotic stability of the zero-dynamics is enough to guarantee the local asymptotic stability of the internal dynamics. Extensions can be drawn to the tracking problem. However, unlike the linear case, no results on the global stability or even large range stability can be drawn for internal dynamics of nonlinear systems, i.e., only local stability is guaranteed for the internal dynamics even if the zero-dynamics is globally exponentially stable.

B.2.6 Control Design

To summarize, control design based on input-output linearization can be made in 3 steps:

1. Differentiate the output y until the input u appears

2. Choose u to cancel the nonlinearities and guarantee tracking convergence

3. Study the stability of the internal dynamics, if $r < n$.

The study of the internal dynamics can be simplified locally by studying that of the zero-dynamics instead. If the zero-dynamics is unstable, different control strategies should be sought.

In the following sections, we will focus on input-output linearization and its application to the tracking control problem.

B.3 Input-Output Linearization for SISO systems

In this section, we discuss input-output linearization for single-input nonlinear systems described by the state space representation

$$\dot{x} = f(x) + g(x)u \quad \text{(B.2)}$$

$$y = h(x) \quad \text{(B.3)}$$

where f , g , h are smooth vector fields and y is the system output. Systems in this form are said to be *linear in control* or *affine*. By input-output linearization, we mean the generation of a *linear* differential relation between the output y and a new input v. Specifically, we shall discuss the following issues:

- How to generate a linear input-output relation for a nonlinear system?
- What are the internal dynamics and zero-dynamics associated with the input-output linearization?
- How to design stable controllers based on input-output linearizations?

For a formal definition of these concepts, we need some mathematical tools.

B.3.1 Mathematical Tools

Definition 2 *A vector function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is called* ***vector field*** *in $\mathbb{R}^n$.*

Definition 3 *A function $h : \mathbb{R}^n \longrightarrow \mathbb{R}$ is called* ***scalar function****.*

Definition 4 *A vector field is said to be* ***smooth*** *if it has continuous partial derivatives of any required order.*

Definition 5 *Given a smooth scalar function $h(x)$ of the state x, the* ***gradient*** *of h is denoted by ∇h*

$$\nabla h = \frac{\partial h}{\partial x}$$

The gradient is represented by a *row-vector* of elements $(\nabla h)_j = \frac{\partial h}{\partial x_j}$.

Definition 6 *Given a smooth vector field $f(x)$ of the state x, the* ***Jacobian*** *of f is denoted by ∇f*

$$\nabla f = \frac{\partial f}{\partial x}$$

It is represented by a $n \times n$ matrix of elements $(\nabla f)_{ij} = \frac{\partial f_i}{\partial x_j}$.

Definition 7 *Let $h : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a smooth scalar function, and $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a smooth vector field on $\mathbb{R}^n$, then the* ***Lie derivative*** *of h with respect to f is a scalar function defined by $L_f h = \nabla h \; f$.*

Thus, the Lie derivative $L_f h$ is simply the directional derivative of h along the direction of the vector f.

Repeated Lie derivatives can be defined recursively

$$L_f^0 h = h$$

$$L_f^i h = L_f \left(L_f^{i-1} h\right) = \nabla \left(L_f^{i-1} h\right) f \qquad \text{for} \;\; i = 1,\; 2, ...$$

Similarly, if g is another vector field, the scalar function $L_g L_f h(x)$ is

$$L_g L_f h = \nabla \left(L_f h\right) g$$

B.3.2 Generating a linear input-output relation

As discussed before, the basic approach of input-output linearization is simply to differentiate the output function y repeatedly until the input u appears, and then design u to cancel the nonlinearity.

Let's place in a region Ω_x in the state space. The process of repeated differentiation means that we start with

$$\dot{y} = \frac{\partial h}{\partial x}\dot{x} = \nabla h\,(f + gu) = L_f\ h(x) + L_g\ h(x)u$$

If $L_g\ h(x) = 0$ for all $x \in \Omega_x$, we can differentiate $\dot{y}$ to obtain

$$\ddot{y} = L_f^2\ h(x) + L_g\ L_f\ h(x)u$$

If $L_g\ L_f\ h(x)$ is again zero for all $x \in \Omega_x$, we shall differentiate again and again,

$$y^{(i)} = L_f^i\ h(x) + L_g\ L_f^{i-1}\ h(x)u$$

until, for some integer r

$$L_g\ L_f^{r-1}\ h(x) \neq 0$$

for some $x = x_0 \in \Omega_x$. Then, by continuity, the above relation is verified in a finite neighborhood Ω of x_0. In this region, the control law

$$u = \frac{1}{L_g\ L_f^{r-1}\ h}\left(-L_f^r\ h + v\right)$$

applied to

$$y^{(r)} = L_f^r\ h(x) + L_g\ L_f^{r-1}\ h(x)u \tag{B.4}$$

yields the simple linear relation

$$y^{(r)} = v$$

The number r of differentiations required for the input u to appear is called the *relative degree* of the system. Based on the above procedure, we can give the following formal definition.

Definition 8 *The SISO system B.2 is said to have* ***relative degree*** r *in a region* Ω *if,* $\forall x \in \Omega$

$$\begin{aligned} L_g\ L_f^i\ h(x) &= 0 \qquad\qquad 0 \le i < r-1 \\ L_g\ L_f^{r-1}\ h(x) &\neq 0 \end{aligned}$$

B.3.3 Normal forms and diffeomorphisms

When the relative degree is $r < n$, the nonlinear system B.2 can be transformed, using $y,\ \dot{y},\ ...y^{(r-1)}$ as part of the new state components, into a so-called "normal form", which shall allow us to take a more formal look at the notions of internal dynamics and zero-dynamics introduced before. Let

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_r \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \\ \vdots \\ y^{(r-1)} \end{bmatrix}$$

the normal form of the system can be written as

$$\begin{cases} \dot{\mu} = \begin{bmatrix} \mu_2 \\ \vdots \\ \mu_r \\ a(\mu,\psi) + b(\mu,\psi)u \end{bmatrix} \\ \dot{\psi} = w(\mu,\psi) \end{cases} \tag{B.5}$$

with the output defined as

$$y = \mu_1$$

The μ_i and ψ_j are referred to as *normal coordinates* or *normal states.* To transform the nonlinear system into the normal form, we need the concept of diffeomorphism.

Definition 9 *A function $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, defined in a region Ω, is called **diffeomorphism** if it is smooth, and if its inverse ϕ^{-1} exists and is smooth.*

Lemma 10 *Let $\phi(x)$ be a smooth function defined in a region Ω in $\mathbb{R}^n$. If the Jacobian matrix $\nabla\phi$ is non-singular at a point $x = x_0$ of Ω, then $\phi(x)$ defines a local diffeomorphism in a subregion of Ω.*

So, we need to construct a diffeomorphism

$$\phi(x) = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_r \\ \psi_1 \\ \vdots \\ \psi_{n-r} \end{bmatrix}$$

such that B.5 is verified. To show that ϕ is a diffeomorphism, it suffices, according to the Lemma, to show that its Jacobian is invertible.

B.3.4 The zero-dynamics

By means of input-output linearization, the dynamics of a nonlinear system is decomposed into an external (input-output) part and an internal ("unobservable") part. Since the external part consists of a linear relation between y and v, it is easy to design the input v so that the output y behaves as desired. Then, the question is whether the internal dynamics will also behave well, i.e., whether the internal states will remain bounded. Since the control design must account for the whole dynamics (and therefore cannot tolerate the instability of internal dynamics), the internal behavior has to be addressed carefully.

The internal dynamics associated with the input-output linearization simply corresponds to the last $(n-r)$ equations $\dot{\psi} = w(\mu, \psi)$ of the normal form. Generally, this dynamics depends on the output states μ. However, as we now detail, we can define an *intrinsic* property of the nonlinear system by considering the system's internal dynamics when the control input is such that the output y is maintained at zero. Studying this so-called *zero-dynamics* will allow us to make some conclusions about the stability of the internal dynamics.

The constraint that the output y is identically zero implies that all of its time derivatives are zero. Thus, the corresponding internal dynamics of the system, or zero-dynamics, describes motion restricted to the $(n-r)$-dimensional smooth surface M_0 described by $\mu = 0$. In order for the system to operate in zero-dynamics, i.e. for the state x to stay on the surface M_0, the initial state of the system $x(0)$ must be on the surface and, furthermore, the input u must be such that y stays at zero, i.e. such that $y^{(r)}(t) = 0$. From B.4 this means that u must equal

$$u_0(x) = -\frac{L_f^r\ h(x)}{L_g\ L_f^{r-1}\ h(x)}$$

Corresponding to this input, and assuming that indeed the system's initial state is on the surface, i.e., that $\mu(0) = 0$, the system dynamics can be simply written in normal form as

$$\begin{cases} \dot{\mu} = 0 \\ \dot{\psi} = w(0, \psi) \end{cases}$$

that is, by definition, the *zero-dynamics* of the nonlinear system B.2.

In normal coordinates, the control input u_0 can be written as a function only of the internal states ψ

$$u_0(\psi) = -\frac{a(0, \psi)}{b(0, \psi)}$$

B.3.5 Tracking control

Consider the problem of tracking a given desired trajectory $y_d(t)$. Let

$$\mu_d = \begin{bmatrix} y_d \\ \dot{y}_d \\ \vdots \\ y_d^{(r-1)} \end{bmatrix}$$

and define the tracking error vector by

$$\tilde{\mu}(t) = \mu_d(t) - \mu(t)$$

We then have the following result:

Theorem 11 *Assume that the system B.2 has relative degree r, that μ_d is smooth and bounded, and that the solution ψ_d of the equation*

$$\dot{\psi}_d = w(\mu_d, \psi_d) \qquad \psi_d(0) = 0$$

exists, is bounded, and is uniformly asymptotically stable. Choose constants

k_i such that the polynomial

$$p^r + k_{r-1}p^{r-1} + ... + k_1 p + k_0$$

has all its roots strictly in the left-half plane. Then, by using the control law

$$u = \frac{1}{L_g \; L_f^{r-1} \mu_1} \left(-L_f^r \; \mu_1 + y_d^{(r)} + k_{r-1}\tilde{\mu}_r + ... + k_0 \tilde{\mu}_1 \right)$$

the whole state remains bounded and the tracking error $\tilde{\mu}$ converges to zero exponentially.

Note that for tracking control to be exact from time $t = 0$ on, using *any* control law, requires that $\mu(0) = \mu_d(0)$.

B.4 MIMO Systems

The concepts used in the above section for SISO systems can be extended to MIMO systems.

In the MIMO case, we consider square systems (i.e., systems with the same number of input and outputs) of the form

$$\begin{aligned} \dot{x} &= f(x) + G(x)u \\ y &= h(x) \end{aligned}$$

where x is the $n \times 1$ state vector, u is the $m \times 1$ control input vector (of components u_i), y is the $m \times 1$ vector of system outputs (of components y_i), f and h are smooth vector fields, and G is a $n \times m$ matrix whose columns are smooth vector fields g_i.

B.4.1 Feedback linearization of MIMO systems

Input-output linearization of MIMO systems is obtained similarly to the SISO case, by differentiating the outputs y_i until the inputs appear. Assume that r_i is the smallest integer such that at least one of the inputs appears in $y_i^{(r_i)}$, then

$$y_i^{(r_i)} = L_f^{r_i} h_i + \sum_{j=1}^{m} L_{g_j} L_f^{r_i - 1} h_i u_j$$

with $L_{g_j} L_f^{r_i-1} h_i(x) \neq 0$ for at least one j, in a neighborhood Ω_i of the point x_0.

Performing the above procedure for each output y_i yields

$$\begin{bmatrix} y_1^{(r_1)} \\ \cdots \\ \cdots \\ y_m^{(r_m)} \end{bmatrix} = \begin{bmatrix} L_f^{r_1} h_1(x) \\ \cdots \\ \cdots \\ L_f^{r_m} h_m(x) \end{bmatrix} + E(x)u$$

where the $m \times m$ matrix is defined obviously.

Define then Ω as the intersection of the Ω_i. If $E(x)$ is invertible over the region Ω, then, similarly to the SISO case, the input transformation

$$u = E^{-1} \begin{bmatrix} v_1 - L_f^{r_1} h_1 \\ \cdots \\ \cdots \\ v_m - L_f^{r_m} h_m \end{bmatrix} \tag{B.6}$$

yields m equations of the simple form

$$y_i^{(r_i)} = v_i$$

Since the input v_i only affects the output y_i, B.6 is called a *decoupling* *control law*, and the invertible matrix $E(x)$ is called the *decoupling matrix* of the system. The system is said to have *relative degree* $(r_1, ..., r_m)$ and the scalar $r_1 + ... + r_m$ is called the *total relative degree* of the system.

An interesting case corresponds to the total relative degree being n. In this case, there is no internal dynamics. With the control law in the form of B.6, we thus obtain an input-state linearization of the original nonlinear system. With the equivalent inputs v_i designed as in the SISO case, both stabilization and tracking can be achieved for the system without any worry about the stability of the internal dynamics. The zero-dynamics of a MIMO system can be defined similarly to the SISO case, by constraining the output to be zero.

The above input-output linearization can be achieved only when the decoupling matrix E is invertible in the region Ω. Given the straightforward procedure used to construct E, sometimes this condition may be rather restrictive.

B.5 Summary

Feedback linearization is based on the idea of transforming nonlinear dynamics into a linear form by using state feedback, with input-state linearization corresponding to complete linearization and input-output to partial linearization. The method can be used for both linearization and tracking control problems, single-input and multiple-input systems. However, the method also has a number of important limitations:

1. it cannot be used for all nonlinear systems

2. the full state has to be measured

3. no robustness is guaranteed in the presence of parameter uncertainty or unmodeled dynamics

Moreover, the input-output linearization lacks systematic global results.

The problem 2 is due to the difficulty of finding convergent observers for nonlinear systems and, when an observer can be found, the lack of a general separation principle (analogous to that in linear systems) which guarantees that the straightforward combination of a stable state feedback controller and a stable observer will guarantee the stability of the closed-loop system.

The problem 3 is due to the fact that the exact model of the nonlinear system is not available in performing feedback linearization. The sensitivity to modeling errors may be particularly severe.

Bibliography

[1] M. Guiggiani, "Dinamica del Veicolo", 1st ed., Città Studi Edizioni, 1998.

[2] G. Burgio, P. Zegelaar, "Integrated vehicle control using steering and brakes", International Journal of Control, Vol. 79, No. 2, February 2006, p.162-169.

[3] J-J. E. Slotine, W. Li, "Applied nonlinear control", Prentice Hall, 1991.

[4] A. Isidori, "Sistemi di Controllo", 2nd ed., Ed. Siderea, 1992.

[5] R. Busch, K. Webers: "Ford Integrated Vehicle Dynamics Control - Concept", Proceedings TÜV Akademie: fahrwerk.tech 2003, March 11-12, 2003

[6] K. Webers, R. Busch: "Ford Integrated Vehicle Dynamics Control - Realization", TÜV, Fahrwerk.tech, München, March 11-12, 2003

[7] M. Sayers: "Standard Terminology for Vehicle Dynamics Simulations", The University of Michigan Transportation Research Institute (UMTRI), February 22, 1996

[8] " Differential (mechanical device)", in Wikipedia: The Free Encyclopedia; (Wikimedia Foundation Inc., updated 27 May 2007, 17:24 UTC) [encyclopedia on-line]; available from

http://en.wikipedia.org/wiki/Differential_(mechanics); Internet; retrieved 30 May 2007.

[9] N.A. Schilke; R.D. Fruechte; N.M. Boustany; A.M. Karmel; B.S. Repa; J.H. Rillings : "Integrated vehicle control", IEEE - International Congress on Transportation Electronics (Convergence), pp. 97-106, October 1988

[10] " Least squares", in Wikipedia: The Free Encyclopedia; (Wikimedia Foundation Inc., updated 3 June 2007, 06:50 UTC) [encyclopedia online]; available from http://en.wikipedia.org/wiki/Least_squares; Internet; retrieved 6 June 2007.