Control Systems
Lesson 2 - Dynamic models

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Dynamic Models - Examples

In this lesson, we examine some examples of dynamic models. Dynamic models are mathematic models.

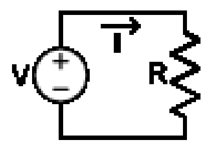
Difference between static and dynamic models: a static model does not account for the element of time, while a dynamic model does. Dynamic models typically are represented with difference equations or differential equations. In order to understand the difference, we can see 3 simple examples of electric circuits.

Models of electric circuits

1. A simple example of static model is given by Ohm's law

$$I = \frac{V}{R}$$

stating that the current through a conductor between two points is directly proportional to the potential difference (i.e. voltage drop or voltage) across the two points, and inversely proportional to the resistance between them.



2 An example of dynamic model is given by the relationship between the time-varying voltage $v(t)$ across an inductor with inductance L and the time-varying current $i(t)$ passing through it:

$$v(t) = L\frac{di(t)}{dt}$$

3 Another example of dynamic model is a capacitor C that is connected to a voltage source $v(t)$. The displacement current is a quantity that arises in a changing electric field. It can occur in a vacuum or in a dielectric medium. In the particular case when it occurs in a vacuum, it does not involve any net linear movement of charged particles. In the case of capacitor, the resulting displacement current $i(t)$ is given by

$$i(t) = C\frac{dv(t)}{dt}$$

Another important difference, in describing mathematic models, is linear vs. nonlinear. Mathematical models are usually composed by variables, which are abstractions of quantities of interest in the described systems, and operators that act on these variables, which can be algebraic operators, functions, differential operators, etc. If all the operators in a mathematical model present linearity, the resulting mathematical model is defined as linear. A model is considered to be nonlinear otherwise.

More formally, a linear function $f(x)$ satisfies the linearity property

$$f(\alpha x_1 + \beta x_2) = \alpha f(x_1) + \beta f(x_2)$$

for any scalars α, $\beta \in \mathbb{R}$, for any vectors x_1, x_2.

An example of linear function is $f(x) = 3x$, while a nonlinear function is $f(x) = \sin x$.

Notation

In the following, we use this notation for time derivatives

$$\begin{aligned} \dot{x} &: = \frac{dx}{dt} \\ \ddot{x} &: = \frac{d^2x}{dt^2} \\ &\vdots \end{aligned}$$

Dynamics of mechanical systems

In many applications we deal with a number of rigid bodies connected to each other in some manner. These connections, called constraints, impose additional conditions on the relative motion of one body with respect to another. Such a constrained set of rigid bodies forms a mechanical system.

The main equation is Newton's second law of motion, stating that "the acceleration of an object is proportional to the force applied, and inversely proportional to the mass of the object".

$$F = ma \quad \text{(scalar relation)}$$

We can also have a vector sum of forces, that is the result of adding two or more vectors together via vector addition.

$$\sum_i \vec{F}_i = m\vec{a} \quad \text{(vector relation)}$$

Remember the acceleration is the first derivative of velocity with respect to time (that is, the rate of change of velocity), or equivalently it is the second derivative of position.

$$a = \dot{v} = \ddot{s}$$

Hence this law provides a dynamic model.

Application 1 - Cruise control

Cruise control (or speed control or autocruise) is a system that automatically controls the rate of motion of a motor vehicle. The driver sets the speed and the system will act in order to maintain the same speed.

At first, we need a mathematic model. As always in modelling, we need a trade-off between simplicity and accuracy.

Assumptions:

- The engine provides a force u in the direction of movement
- We neglect moments of inertia for wheels
- Friction is proportional to longitudinal velocity and reducing (opposing) the speed of the vehicle.

Now we set the coordinate system. The x-axis is chosen in order to have positive distance towards the right.

So we have

- x is the position
- $\dot{x}$ is the velocity
- $\ddot{x}$ is the acceleration
- u is the force provided by the engine and it is positive with respect to the system coordinates
- The friction force F_f is proportional to longitudinal velocity and negative, with proportionality constant referred as b, so we have

$$F_f = -b\dot{x}$$

The law of motion so takes the particular form

$$u - b\dot{x} = m\ddot{x}$$

If the quantity of interest is position, we can write it as

$$\ddot{x} = -\frac{b}{m}\dot{x} + \frac{1}{m}u$$

i.e. a second order linear differential equation.

If the quantity of interest is velocity, we can define

$$v := \dot{x}$$

so we have

$$\dot{v} = -\frac{b}{m}v + \frac{1}{m}u$$

i.e. a first order linear differential equation.

This completes the modelling of the system, that can be regarded as the first step in solving a control problem. Techniques for solving problems like this will be explained later in this course.

Application 2 - Model of suspension

In vehicles, suspensions are linkages that connect a vehicle to its wheels. Suspension systems serve a dual purpose – contributing to the car's handling and braking for good active safety and driving pleasure, and keeping vehicle occupants comfortable and reasonably well isolated from road noise, bumps, and vibrations. These goals are generally at odds, so the tuning of suspensions involves finding the right compromise. The suspension also protects the vehicle itself and any cargo or luggage from damage and wear. The design of front and rear suspension of a car may be different.

We consider the simple "quarter-car model", meaning that a quarter of the weight of the car rests on each wheel. The mass of each wheel is denoted with m_1 while, if the mass of the car is M, the mass acting on each wheel is $m_2 = \frac{M}{4}$.

The suspensions and the tyres are flexible objects, so thay can be simply modelled with two springs. A spring has the ability for storing mechanical energy. The simplest model of spring obeys Hooke's law, which states that the force with which the spring pushes back is linearly proportional to the distance from its equilibrium length

$$\vec{F} = -k\vec{x}$$

where

- $\vec{x}$ is the displacement vector - the distance and direction in which the spring is deformed

- $\vec{F}$ is the resulting force vector - the magnitude and direction of the restoring force the spring exerts

- k is the spring constant or force constant of the spring.

The model of suspension also contains a damper, that is a sort of shock absorber reducing the effect of traveling over rough ground, leading to improved ride quality. It can be represented by a piston, and it has the property of opposing velocity of variation of the relative displacement between two masses, that are the wheel m_1 and the "quarter of car" m_2. We consider the damper force to be proportional (in accord with a proportionality constant b) with respect to velocity of variation of the relative displacement between m_1 and m_2.

Experimental identification of parameters

The identification of parameters is an important phase in control systems design, usually following the modelling. It is a problem consisting of taking the measured data and producing an estimate of the parameters.

In the current example, we define k_s as the spring constant of the suspension, modelled as a spring, and k_w as the spring constant of the wheel enclosed by the tyre. By using the model of Hooke's law, for example, a person can sit on the wheel enclosed by the tyre, and the displacement can be measured. For example, for a person weighing 100 kg sitting on the wheel, the wheel shows 1 mm displacement. Hence

$$k_w = \frac{F}{x} = \frac{mg}{x} = \frac{1000}{0.001} = 10^6 \ N/m$$

Similarly we can estimate $k_s = 1.3 \times 10^5 \ N/m$ and, using a different model, the damping coefficient b.

Modelling

We set the frame of reference, denoting with x and y the spatial coordinates of the two masses. These equilibrium positions are different from the ones observable putting springs horizontally. This constant displacement occurs because of the gravity force acting on them. So, if we define the points $x = 0$ and $y = 0$ in the equilibrium positions, we can disregard gravity in writing the model.

Notice that the relative position (displacement) between the two masses is $y - x$, while the relative velocity is $\dot{y} - \dot{x}$.

If r is a point on the road surface (constant if the road has a plane surface, otherwise it is a function $r(t)$), by using free body diagrams, we can derive the equations for the two masses

$$\begin{aligned} b(\dot{y} - \dot{x}) + k_s(y - x) - k_w(x - r) &= m_1\ddot{x} \\ -k_s(y - x) - b(\dot{y} - \dot{x}) &= m_2\ddot{y} \end{aligned}$$

Rearranging the terms, we have

$$\begin{aligned} \ddot{x} &= \frac{b}{m_1}(\dot{y} - \dot{x}) + \frac{k_s}{m_1}(y - x) - \frac{k_w}{m_1}x + \frac{k_w}{m_1}r \\ \ddot{y} &= -\frac{b}{m_2}(\dot{y} - \dot{x}) - \frac{k_s}{m_2}(y - x) \end{aligned}$$

Newton's Law for rotation

In order to apply Newton's law for unidimensional rotation, instead of translation, we have to make the following modifications

- the (resulting) force F (expressed in N) is substituted by the (resulting) moment M of external forces (expressed in Nm), or torques, about the centre of mass;
- the mass m (expressed in kg) is substituted by the moment of inertia I (expressed in $kg * m^2$) about the centre of mass;
- the linear acceleration a (expressed in m/s^2) is substituted by the angular acceleration α (expressed in rad/s^2).

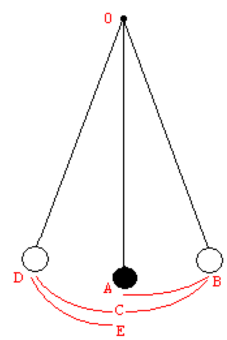
Finally, we have

$$M = I\alpha.$$

Application 3 - Simple gravity pendulum

A pendulum is a mass that is attached to a pivot, from which it can swing freely. This object is subject to a restoring force due to gravity that will accelerate it toward an equilibrium position. When the pendulum is displaced from its place of rest, the restoring force will cause the pendulum to oscillate about the equilibrium position.

A basic example is the simple gravity pendulum or bob pendulum. This is a mass (or bob) on the end of a string of negligible mass, which, when initially displaced, will swing back and forth under the influence of gravity over its central (lowest) point.



Modelling

Differently from the previous examples, it is an example of nonlinear system.

The pendulum is an example of rotating body where the pivot is in a fixed position (with respect to an inertial frame of reference). So we can consider M and I about the pivot.

The moment of inertia is

$$I = ml^2$$

where l is the length of the string. The angle which the pendulum swings through is referred as θ. An applied external torque is denoted with T_c, positive if the rotation is anticlockwise.

The motion equation is

$$\begin{aligned} T_c - mgl\sin\theta &= I\ddot{\theta} \\ T_c - mgl\sin\theta &= ml^2\ddot{\theta} \end{aligned}$$

that can be written in the "normal form"

$$\ddot{\theta} = -\frac{g}{l}\sin\theta + \frac{1}{ml^2}T_c$$

having a nonlinear second order differential equation. For small angles, we can approximate

$$\sin\theta \approx \theta$$

so we have the linear second order differential equation

$$\ddot{\theta} = -\frac{g}{l}\theta + \frac{1}{ml^2}T_c$$

If no external torques are applied, we have the simple harmonic motion, i.e. the motion of a simple harmonic oscillator that is neither driven nor damped

$$\ddot{\theta} + \omega_n^2\theta = 0$$

where $\omega_n^2 = \frac{g}{l}$, whose solution is the displacement

$$\theta\left(t\right) = A\cos\left(\omega_n t + \phi\right)$$

The period of the oscillation is

$$T = \frac{1}{f} = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{l}{g}}$$

Preview 1 - Concepts of systems and control

In control systems, we deal with dynamical systems. They are representations of physical, biological or information models. Dynamical systems theory provides methods for the analysis of systems, while the control theory deals with the synthesis of controllers, that are part of the dynamical systems.

Roughly speaking, in dynamical systems, we distinguish input variables, output variables and state variables.

- The inputs are the variables that influence a process and can be modulated in a controlled way.
- The outputs denote the signals that we are interested in.
- The state is a collection of variables that completely characterizes the motion of a system.

The desired output of a system is called the reference. When one or more output variables of a system need to follow a certain reference over time, a controller manipulates the inputs to a system to obtain the desired effect on the output of the system. That is the main goal of control theory.

Preview 2 - Transfer function representation of dynamical systems

Transfer functions are a method of representation of dynamical system in the frequency domain. This approach differs from the state-space representation, also known as the "time-domain approach".

The transfer function representation is only for linear time-invariant systems (LTI).

Linearity means that the relationship between the input and the output of the system satisfies the superposition property. If $y_1(t)$ is the output resulting from the sole input $u_1(t)$ and $y_2(t)$ is the output resulting from the sole input $u_2(t)$, applying as input to the system the signal

$$u(t) = c_1u_1(t) + c_2u_2(t)$$

where c_1 and c_2 are constants, then the output of the system will be

$$y(t) = c_1y_1(t) + c_2y_2(t)$$

The property can be extended to more than 2 signals. If $y_k(t)$ is the output resulting from the sole input $u_k(t)$, if the input of the system is

$$u(t) = \sum_{k=1}^{n} c_k u_k(t)$$

where c_k are constants for all k, then the output of the system will be

$$y(t) = \sum_{k=1}^{n} c_k y_k(t)$$

Time invariance means that whether we apply an input to the system now or T seconds from now, the output will be identical, except for a time delay of the T seconds. If the output due to input $u(t)$ is $y(t)$, then the output due to input $u(t-T)$ is $y(t-T)$. More specifically, an input affected by a time delay should effect a corresponding time delay in the output, hence time-invariant.

LTI systems can be characterized in the frequency domain by the system's transfer function, which is the Laplace transform of the system's impulse response. In the frequency domain, the system (and each time function) is no longer characterized with a real argument t ($t \geq 0$), denoting the time variable, but with a complex argument s

$$f(t) \iff F(s)$$

Laplace transform will be explained later in this course, however it is possible to compute the transfer functions of systems, applying the rule that in the dynamic equation, the general time derivative $f^{(k)}(t)$ of a function $f(t)$ has to be replaced by its transformed term, according to this equivalence

$$\begin{array}{ccc} f(t) & \iff & F(s) \\ \dot{f}(t) & \iff & sF(s) \\ & & \vdots \\ f^{(k)}(t) & \iff & s^k F(s) \end{array}$$

The ratio

$$P(s) := \frac{Y(s)}{U(s)}$$

is called transfer function.

Transfer function for the cruise control application

In the example of cruise control, the dynamic equation is

$$\dot{v} = -\frac{b}{m}v + \frac{1}{m}u$$

The input $u(t)$ is the force provided by the engine.

The quantity of interest for the control problem is velocity, so we say the output is

$$y(t) = v(t)$$

and the dynamic equation is written as

$$\dot{y}(t) = -\frac{b}{m}y(t) + \frac{1}{m}u(t)$$

We consider the equivalence

$$u(t) \iff U(s)$$

$$\begin{array}{ccc} y(t) & \iff & Y(s) \\ \dot{y}(t) & \iff & sY(s) \\ \ddot{y}(t) & \iff & s^2 Y(s) \end{array}$$

so we have, in the frequency domain

$$\begin{aligned} sY(s) &= -\frac{b}{m}Y(s) + \frac{1}{m}U(s) \\ \left(s + \frac{b}{m}\right)Y(s) &= \frac{1}{m}U(s) \end{aligned}$$

Hence we conclude that the transfer function is

$$P(s) = \frac{Y(s)}{U(s)} = \frac{\frac{1}{m}}{\left(s + \frac{b}{m}\right)}.$$

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